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THE GROWTH AND VARIABILITY OF INTELLIGENCE

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THE GROWTH AND VARIABILITY OF INTELLIGENCE

PART I

BY C. A. RICHARDSON, M.A.

PART II

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AND C. W. STOKES, M.A.

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SYNOPSIS

PART I

PART II

PREFATORY NOTE

DUE and grateful acknowledgment is made in the text following to all who helped to make possible the investigation the results of which are to be described. But the authors are happy to avail themselves of this opportunity of expressing their thanks to the *British Psychological Society* for the financial grant made by the Society towards the expenses of publication of this monograph.

C. A. R.
C. W. S.

PART I

THE GROWTH AND VARIABILITY OF INTELLIGENCE

BY C. A. RICHARDSON

VARIOUS methods have from time to time been proposed for plotting a curve which shall accurately represent the growth of intelligence in children. One of the earliest and most obvious of these was to apply an intelligence test to a large and 'random' sample of children, and from its results to plot the curve of norms giving the mean scores in successive years. Such a curve provides an indication of the rate of growth of intelligence. It can show with some truth whether the growth is on the whole continuous or spasmodic and irregular, and it can also indicate when, on the average, intelligence reaches maturity. Apart from this, however, it reflects but imperfectly the facts that are being sought. The reason for this imperfection is that the units of measurement used in plotting the curve are the marks scored in the test, and we do not know the absolute value of these units in terms of intelligence nor can we assume that all the units are of equal value. On the contrary we may be quite certain that, in general, they are *not* of equal value. The difference of 10 marks between the scores of (say) 50 and 60 in a test will not in general be equal in terms of intelligence to the difference of 10 between the scores of (say) 80 and 90. In most (probably in all) cases the second difference will represent a greater divergence on the absolute scale of intelligence than the first difference. The larger a given score, the greater must be the rise in intelligence level necessary to increase that score by a given amount. The more marks obtained, the more difficult it is, in general, to increase the score by one more mark. The units are becoming larger and larger in absolute value. Moreover, we cannot even be sure that equal scores represent equal levels of intelligence. For such scores will nearly always be made up of different selections of test items correctly answered, and the difficulty of one set may not be equal to that of another set equal in number to the first.

Evidently, then, it is necessary to consider each test item separately and attempt to express its degree of difficulty somehow in terms of the amount of intelligence required to answer it correctly. A method

of doing this has been proposed and applied practically by Thurstone (see the *Journal of Educational Psychology*, October 1925 and November 1929). It would not be possible, nor is it necessary, to describe the method in detail here. But it may be outlined briefly. It is first necessary to ascertain the percentage of children in each age-group who answer a given test item correctly. This percentage is obtained for every item and for every age-group. It is assumed that the distribution of intelligence is normal in each age-group. In each case the position of the ordinate of the normal distribution curve which cuts off a percentage of the total area under the curve equal to the percentage success in the particular item by the particular age-group is then determined. This position is expressed as a distance from the mean ordinate in terms of the standard deviation (σ). It is thus possible to assign to every item a 'σ value' for each age-group in turn. If the percentage success exceeds 50, the σ value is reckoned negative; if it is less than 50, the σ value is reckoned positive. Let p, q be any two age-groups, and let n be the number of 'overlapping' items for the two groups, i.e. items each of which is answered by *some, but not all*, of the children *in each of the groups*. Let $X(p)$ be the σ value of an item for group p, and $X(q)$ the σ value of the same item for group q. Then if $M(p)$ and $M(q)$ be the mean 'test intelligences' and $\sigma(p)$ and $\sigma(q)$ the standard deviations in test intelligence of the respective age-groups, expressed in the units of an absolute scale, it can easily be shown that

$$\sigma(q) = \frac{S(p)}{S(q)}\,\sigma(p) \qquad \text{......(A),}$$

and

$$\begin{aligned} M(q) &= \sigma(p)\left[m(p) - \frac{S(p)}{S(q)}\,m(q)\right] + M(p) \\ &= \sigma(p)\,m(p) - \sigma(q)\,m(q) + M(p) \qquad \text{......(B),} \end{aligned}$$

where

$$m(p) = \frac{\Sigma X(p)}{n},\quad m(q) = \frac{\Sigma X(q)}{n},\quad S(p) = \sqrt{\frac{\Sigma X^2(p)}{n} - m^2(p)},$$

and

$$S(q) = \sqrt{\frac{\Sigma X^2(q)}{n} - m^2(q)},$$

the summation denoted by the Σ's being taken for the n 'overlapping' items*. Thus it is possible to obtain the mean intelligence and the standard deviation in intelligence (expressed on an absolute scale) of

* A short statement of the proof of formulae (A) and (B) is given in Appendix III to Part I.

any age-group from those of any other age-group. In actual practice adjacent age-groups are usually selected for this purpose. We start by taking the mean of one of the age-groups as the zero point or *origin* of measurement, and the standard deviation of this (or another) age-group as the *unit* of measurement. From the data given by the percentage successes in the test items, we can then proceed from one age-group to the next by means of equations (A) and (B) until we have expressed the mean intelligence and the standard deviation of each age-group in terms of the standard deviation of the selected group as unit, the mean intelligence being measured in each case from that of the selected group as zero. In so far as the origin is chosen thus arbitrarily, the scale is not strictly an 'absolute' one; but, as will appear in the sequel, it may be possible to determine, in the result, an absolute zero for the scale. It must be pointed out that the method briefly described in the foregoing can only be applied when a certain statistical condition is satisfied. The criterion for this is that the σ values (X's) of the test items for any one age-group should correlate perfectly ($r = 1$) with the σ values for any other age-group. This condition is never exactly satisfied, but in practice it may be regarded as sufficient if the correlation between the σ values of the items for each pair of *adjacent* age-groups is of the order 0·98.

Instead of scaling each test item in the way indicated, Thurstone's method may be applied by scaling the actual scores in the test and assigning to each score a σ value for each age-group according to the percentage of children in the age-group who reach or exceed the score in question. When all the σ values have been assigned the procedure is the same as before, the summations covering overlapping scores (i.e. scores reached or exceeded by *some but not all* of the children in each of the pair of age-groups in question), instead of overlapping items. But it would seem, at any rate *prima facie*, that the 'item' method is more precise in principle than the 'score' method for a reason already indicated, namely that a given score can be made up in any number of different ways. Hence there is an ambiguity associated with making a given score which does not arise when we are considering success or failure in answering a given item.

The investigation to be described was undertaken as an experiment in the application of Thurstone's method on a large, and statistically complete, scale, in order, firstly, to obtain a reliable curve of mental growth between the ages included; and, secondly, to discover whether certain special results of the experiment would agree

with those obtained by Thurstone. The actual test items were scaled, not the raw scores. Perhaps the chief claim that can be made for the experiment is the completeness of the 'population' on which it was carried out. This population consisted of *all* the children in a large industrial town between certain age-limits—some 12,000 in all. To the best of the writer's knowledge no other investigation directed to this end has been carried out on this scale and with this degree of completeness. In some of the American inquiries the total numbers were considerably larger, but the results seem to have been collected from a number of different areas and then pooled.

On a fixed day, as far as possible all the children in the town of Blackburn, over the age of 6 and under the age of 14, were given a group intelligence test (known as the '*K*' test*) specially drawn up by the writer for the purpose. The children took the test in their own schools, the latter including all the Secondary, Elementary and Central, Private, and Special (for defectives) Schools of the borough. It is safe to say that very few of the whole child-population within the age-range escaped the net—only those, in fact, who were absent from school on that day for sickness or other reasons.

One of the chief difficulties of the experiment was the necessity for using a group test, owing to the large number of children involved. It was by no means easy to devise a single group test which would be reasonably suitable for all children of 6 to 13 years of age; but without such a test the experiment could not have been satisfactorily carried out. The test finally drawn up consisted of 100 items. In the preliminary trials and standardizing an attempt was made to ensure that the test should include, for each age-group, at least six or eight items which would be answered with approximately 50 per cent. success by the group (as well as questions of intermediate difficulty), together with a number of items easier than the median 6-year-old items, and others more difficult than the median 13-year-old items. As the preliminary trials were necessarily limited in scope, it was extremely difficult to ensure that the final results would work out in this way. But, on the whole, the outcome was satisfactory. It will be described in full detail later. It is enough to mention here that the test proved to be somewhat overweighted at the 'difficult' end. That the inclusion of a considerable number of hard questions was necessary, was shown by the fact that nearly 50 children scored

* This test has now been published by Messrs Harrap under the title of the *Simplex Junior Group Intelligence Scale.*

between 90 and 100 on the test. But there was no corresponding 'make-weight' at the other end in the form of very easy questions below 6-year-old median level, owing to the great difficulty of finding items sufficiently easy and at the same time suitable for inclusion in a group test of the kind necessarily employed.

A copy of the *K* test is given in Appendix I to Part I. It will be seen that it begins with easy number questions followed by easy language questions of the 'completion' type. Then comes a section on objects of common observation and knowledge. There follow, in order, Rhymes, Directions, Vocabulary, Number Series, Analogies, and, finally, various types of Reasoning. The three tests of the 'multiple-choice' type (Rhymes, Vocabulary and Analogies) were so framed as to reduce the effect of random answering as far as possible, and the average effect of guessing would be to add not more than $3\frac{1}{2}$ per cent. to the combined score on these three tests. The reliability of the whole scale, as determined from a group of 95 unselected 10-year-olds, was 0·95, the standard deviation of the scores of this group being 14. This gives a standard error of score of 3·1. The validity of the test, determined from the same group against a criterion consisting of a very careful set of teacher-estimates, was 0·927—an unusually high value.

At the conclusion of the test the papers were marked by the Heads of the schools and their staffs, who willingly undertook what was a very considerable labour—considerable, for in addition to finding the distributions and mean scores in each age-group, it was necessary to determine, again for each age-group, the number of children who correctly answered each of the test items. A group of colleagues then helped the writer to combine the results from the various schools, and the percentage success of each complete age-group in each item was thus obtained, together with supplementary information of interest supplied by the norms and distribution tables of raw scores for the different age-groups. The figures for percentage success were then converted to σ values, and Thurstone's method applied to the results.

The results of the test, in terms of the raw scores, are set out in the three following tables. Table I gives the mean scores and the standard deviations for each age-group; Table II contains the distributions; and Table III gives the approximate values of the medians and quartiles. A scrutiny and comparison of these tables will show that the scores were distributed with a reasonably close approach to normality except in the extreme age-groups.

Table I

Norms and standard deviations

Age	Norm	Standard deviation
6+	11·3	10·2
7+	23·2	12·7
8+	32·0	12·9
9+	39·6	14·0
10+	45·3	14·8
11+	53·2	15·0
12+	59·3	14·9
13+	63·6	16·2

Table II

Distributions

Age	Score										Total
	0–9	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89	90–100	
6+	565	295	207	46	14	1	—	—	—	—	1128
7+	263	298	435	305	127	20	2	—	—	—	1450
8+	100	177	415	544	332	118	15	—	—	—	1701
9+	38	117	260	542	507	327	118	19	1	—	1929
10+	25	63	133	279	436	450	197	54	8	—	1645
11+	9	19	37	120	226	314	237	99	24	3	1088
12+	4	13	18	53	142	251	270	164	58	4	977
13+	12	10	26	54	144	276	340	296	152	38	1348
										Grand total	11266

Table III

Medians and quartiles

Age	Lower quartile	Median	Upper quartile
6+	5	10	19
7+	13	23	33
8+	23	33	41
9+	31	40	50
10+	37	47	57
11+	44	54	64
12+	51	60	69
13+	53	65	75

The line of norms is given in Fig. 1.

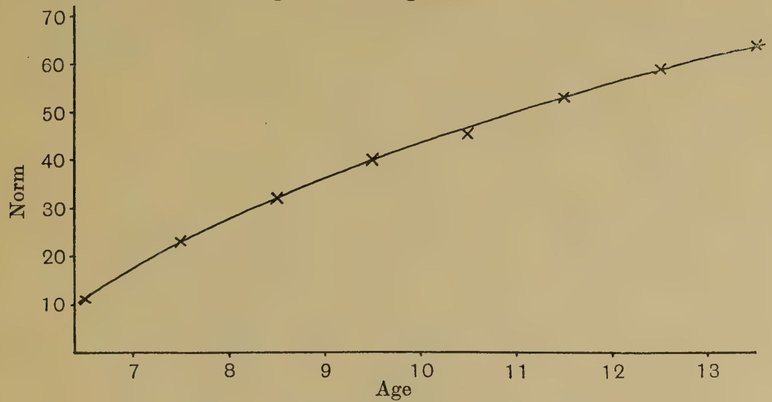


Fig. 1. *K* test raw scores. Line of norms.

So far the results call for little comment. They are generally similar to those associated with most well standardized group intelligence tests. Minor details are the slight dip in the norm at 10 + and the rather noticeable increase in 'spread' in the 13-year-old group. The latter is not, however, extremely marked, and is probably due largely to accidental circumstances—it is rather curious, for example, that as many as twelve children in this group scored less than 10 marks.

It will be seen that the line of norms shows definite signs of negative acceleration, but whether this really reflects a gradual slowing down in mental growth cannot be decided until we have considered the result of applying the method of absolute scaling to the test items.

In passing on to the absolute scaling of the test results*, it should first of all be remarked that the statistical criterion which justifies the scaling was well satisfied. The correlations between the σ values of the items for pairs of adjacent age-groups are given in Table IV.

Table IV

Age-groups	Correlations
6–7	0·976
7–8	0·987
8–9	0·986
9–10	0·973
10–11	0·986
11–12	0·995
12–13	0·997

* The complete sheet of data obtained from the test will be found at the end of this monograph.

The values of M, the mean performance, and σ, the standard deviation, for the various age-groups, which were obtained by absolute scaling, are set out in Table V. They are measured from the mean of the 7-year-old group as origin, and expressed in terms of the standard deviation of that group as unit.

Table V

Age	M	σ
6+	−0·773	0·9712
7+	0	1
8+	0·4468	1·0795
9+	0·9002	1·158
10+	1·1691	1·231
11+	1·5908	1·272
12+	1·9170	1·312
13+	2·1987	1·424

A check on these figures was made in the following way. The σ values of the test items for the pairs of adjacent age-groups were plotted on squared paper, and, on the basis of a careful scrutiny of the seven plots thus obtained, a selection of 'best' items was chosen. Sixty-five items were included in this selection, and all the calculations were repeated for these. The results are given in Table VI.

Table VI

Age	M	σ
6+	−0·7965	0·9993
7+	0	1
8+	0·4214	1·04
9+	0·8821	1·121
10+	1·1857	1·188
11+	1·6202	1·248
12+	1·9218	1·272
13+	2·2138	1·365

The most striking thing about the figures in this table is their close agreement with those obtained from the full scale and given in Table V. In most cases the difference between corresponding figures is not more than from 2 to 3 per cent., and in no cases does it exceed about 5 per cent.

One point of interest emerges from the selection of those items which the statistical criterion indicated as the best. Some evidence as to the comparative suitability of the various types of question in this particular test-scale is provided by a consideration of the

percentage of items of different kinds which was retained in the selected group. These percentages are given in descending order in Table VII.

Table VII

Type of question	Percentage retained in Select List
Directions (Nos. 50–59)	100
Number series (Nos. 71–79)	100
Rhymes (Nos. 42–49)	88
Reasoning (3-dimensional relations—Nos. 96–100)	80
Reasoning (Ages—Nos. 92–95)	75
Analogies (Nos. 80–86)	71
Vocabulary (Nos. 60–70)	55
'Information' (Names—Nos. 29–41)	54
Easy number (Nos. 1–17)	47
Reasoning (Money—Nos. 87–91)	40
Easy language (Completion—Nos. 18–28) ...	36

Re-arranged in order of consecutive items, the list is as follows:

Table VIII

Type of question	Percentage retained
Easy number (Nos. 1–17)	47
Easy language (Nos. 18–28)	36
'Information' (Nos. 19–41)	54
Rhymes (Nos. 42–49)	88
Directions (Nos. 50–59)	100
Vocabulary (Nos. 60–70)	55
Number series (Nos. 71–79)	100
Analogies (Nos. 80–86)	71
Reasoning (Money—Nos. 87–91)	40
Reasoning (Ages—Nos. 92–95)	75
Reasoning (3-dimensional relations—Nos. 96–100)	80

Too much importance should not be attached to these figures. But the comparative excellence of questions of the Number series and Directions types is at any rate clear, all questions of these kinds being retained. The comparative poorness of the Easy number, Easy language, and Reasoning (Money) questions is equally obvious, less than half of these types of question being retained. Their relative unsuitability is almost certainly at least as much due to the actual material of the questions as to their type, especially in the case of the Reasoning (Money) questions (Nos. 87–91). But it is in any case necessary to emphasize that the unsuitability of some of the questions is only *comparative*. The high value of the correlations between the

σ values of items for adjacent age-groups is sufficient evidence that very few of the questions were 'poor' in an absolute sense, and this conclusion was amply confirmed by inspection of the plots of the σ values. The scrutiny on which certain items were thrown out and others retained, was, as a matter of fact, particularly drastic.

Attention may here be drawn to the fact, noted by Thurstone, that, if the scale we are considering is truly absolute, and if the test employed is accurate, the scale values obtained from the various age-groups for the same items should be nearly (theoretically, exactly) equal. The writer did not attempt to compute the scale value of every item from each age-group, but he found the differences in the scale values obtained from two extreme groups, the 7-year-old and 13-year-old, for the 92 overlapping items of the full scale for those groups, as the comparison will in general be most unfavourable for widely separated age-groups. For the purposes of such a comparison it is obviously necessary to express the scale values as measured from the *absolute zero* of the scale, to which reference is made later. The greatest *percentage* difference in the scale values of an item obtained from the two age-groups mentioned was 19 and the least was zero, while the mean percentage difference was only 4·6, the mean variation about this being but 2·6. This is irrespective of the sign of the differences. Taking the latter into account, the mean percentage difference was 0·04 and the distribution of the differences about this mean was fairly symmetrical, the mean variation on either side of it being 2·4 per cent.

We may now return to consider the figures for M and σ given in Table V. Only one of these figures is unexpected, namely the value of M for the 6-year-old group. This is very low, and the result is confirmed by Table VI. As, however, the test is evidently difficult for this group, most of the σ values of the items being high, and therefore to a corresponding extent unreliable, the calculations for this age-group were again repeated, but this time making use of Thurstone's more refined method with weighted observations*. Nevertheless the results were practically identical with those already obtained, being $M(6) = -0{\cdot}769$, $\sigma(6) = 0{\cdot}996$. Evidently, then, there is either a distinct drop in the intelligence of the 6-year-old group, or, perhaps, the novelty of the situation, or the relative unsuitability of the earliest test items, or both, prevented the 6-year-olds from doing themselves full justice. We shall see in Part II that $M(6)$, at any rate,

* See *Journal of Educational Psychology*, October 1928.

is probably accurate, but for the moment we will ignore these rather doubtful figures for the 6-year old group.

The values for M at different ages, together with the curves of performance at $+1\sigma$ and -1σ above and below the mean respectively, are plotted in Fig. 2.

It will be observed that the curve for M, if unsmoothed, would show a slight kink towards the age of 11 years. The reason for this will be fully investigated in Part II, where we shall also see that, if there is a definite point of inflexion on the M curve, it probably comes much earlier than Thurstone's estimate of about $9\frac{1}{2}$ years. As regards later ages, on the other hand, the *probable* course of the M curve after

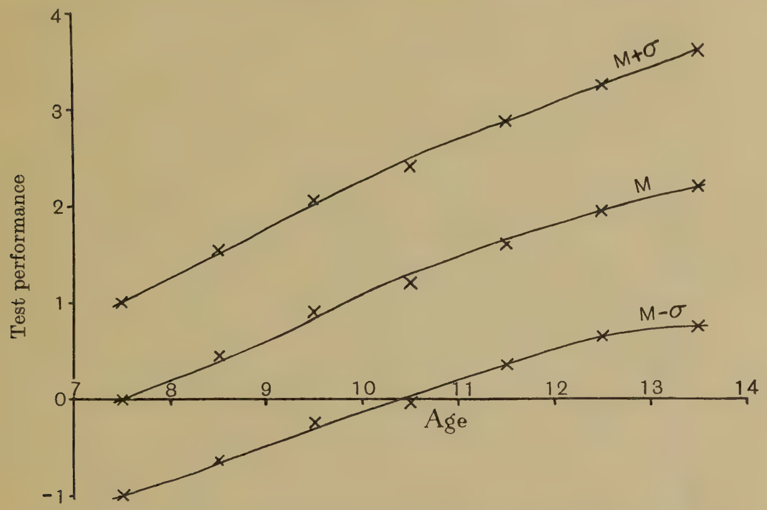


Fig. 2. Mean growth curve of intelligence on the absolute scale given by K test, and curves at ± 1 standard deviation.

14 years is considered in Part II. But really conclusive evidence as to the age at which intelligence reaches maturity will not, of course, be available until it is possible to obtain for test large random samples up to 18 or 20 years of age. Thurstone's samples for the older groups do not seem to be at all satisfactory.

The curves in Fig. 2 also show clearly the increase in the 'spread' of intelligence with increasing age, which is in conformity with Thurstone's conclusions.

A more interesting and important result is obtained from Table V by making a plot to show the relationship between M and σ. This plot is given in Fig. 3.

Neglecting the point corresponding to the 6-year-old group, the plot in Fig. 3 exhibits a reasonably close approach to linearity. This confirms Thurstone's most striking conclusion, namely, that the 'spread' in intelligence at any age is proportional to the mean level of intelligence at that age, or $\frac{\sigma}{M}$ = constant, where M is measured, not from the arbitrary origin chosen, but from an absolute zero located by producing the σ, M line to cut the line $\sigma = 0$. In all that follows it will be assumed that M is now measured from this absolute zero, which will be found to be at the point $-5{\cdot}56$, that is $5{\cdot}56\sigma\,(7)$,

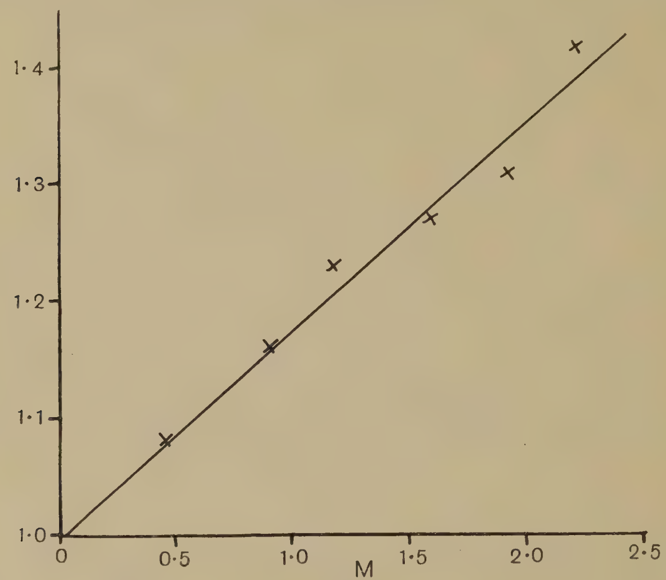


Fig. 3. Plot of σ against M—absolute scale (K test).

below our arbitrary origin $M\,(7)$. Let us denote this constant value of $\frac{\sigma}{M}$ by k. Now if absolute scaling be applied to the results of different tests of intelligence, or even to the same test scaled in different ways (e.g. by items or by raw scores), we should not expect to get the same set of values for σ and M in each case. The situation is, indeed, somewhat analogous to that of the measuring of temperature by different kinds of thermometers. The 'absolute' scale obtained for any given test is absolute so far as that test is concerned in the sense that the units have the same value at all points of the scale, but it is not absolute in the sense that it is identical for all tests. On the

other hand, if the various intelligence tests measure something definite, and *if they all measure the same thing*, we *should* presumably expect the value of $k \left(= \frac{\sigma}{M}\right)$ to be the same for all tests. The value of k given by Thurstone is of no use for our present purpose, as his 'population' was admittedly highly selected. But fortunately evidence on the point is provided in an article by C. L. Odom in the *Journal of Educational Psychology* for September 1929, in which are given the results of the absolute scaling (by raw scores) of a number of well-known group tests of intelligence. Seven such tests are considered, but we shall disregard the results for two of these as the populations on which they were based were not sufficiently satisfactory. In the case of the remaining five, none of the populations was a complete sample in the sense in which the population in the present study was complete, and although the data for one of the tests (National Intelligence Scale A) were obtained from more than twice the number of cases tested in the writer's investigation, these cases were obtained from nineteen different communities. Hence it was to be expected that the plots of the values of σ and M obtained by Odom in scaling the tests he used would be more irregular than that obtained from the K test, and this expectation was in fact confirmed. Nevertheless it was possible, without much difficulty, to obtain a best fitting straight line for the data in each case, on the assumption that the relation between σ and M is really linear. The resulting values of k are given in Table IX.

Table IX

Test	k
Dearborn I	0·13
Dearborn II	0·15
Otis Advanced	0·14
National A	0·20
Illinois (Bloomington data)	0·24
	Mean 0·17

As a matter of interest, the results of the K test were again reduced to an absolute scale, but this time on the basis of raw scores. The results are given in Table X, but they must be regarded only as approximations, as the grading of the raw scores was relatively coarse, namely by intervals of 10.

Table X

Age	M	σ
6+	−0·988	0·999
7+	0	1
8+	0·688	0·995
9+	1·343	1·123
10+	1·770	1·237
11+	2·410	1·347
12+	2·831	1·387
13+	3·265	1·487

For this scaling $k = 0{\cdot}17$. For the original absolute scaling of the K test, namely by the consideration of individual items, it is clear from Fig. 3 that $k = 0{\cdot}18$.

The values of k obtained from the various tests are not conclusive one way or another, but they are perhaps sufficiently similar in the circumstances to constitute presumptive evidence that all the tests are measuring something definite which is the same in each case. Moreover, it is possibly significant that in the case of the K test used in the present investigation, the only test of all those considered which was applied to a complete sample, the value obtained for k is practically identical with the mean of the values obtained from the results of the five group tests investigated by Odom.

The par age values for the items of the K test will be found, in the cases where they can be calculated, in Appendix II to Part I. The par age for an item is the age at which it is correctly answered by exactly 50 per cent. of the children. The approximate position of each item on the absolute scale is also given in Appendix II to Part I.

The writer will not attempt to draw any dogmatic conclusions from the investigation he has described, even were such possible. But he feels that he has perhaps obtained some evidence

(1) To support Thurstone's conclusions that:

(*a*) The absolute variability of intelligence increases with age; and

(*b*) The spread of intelligence at any age is proportional to its mean level at that age.

(2) To indicate that all well devised intelligence tests probably measure something definite which is the same in each case.

(3) To establish the probable constant value of $k \left(= \frac{\sigma}{M}\right)$ required by (2) above, as about 0·18.

As a by-product, a very thoroughly standardized group intelligence test has been obtained, fairly suitable for all ages from 7 years to 14 years.

In conclusion, the writer would like to express his grateful thanks to the officials of the Blackburn Local Education Authority who made the investigation possible, to the Heads and assistant staffs of the schools for administering and scoring the tests, and tabulating the results; to the Heads of schools in neighbouring areas who assisted in the preliminary trials and in the determination of the reliability and validity of the test; and to a number of colleagues who provided certain of the test items or helped in the collation of the results sent in from the schools.

APPENDIX I

THE 'K' TEST

See these sums:

Add $4 + 1 = 5$
Take away $3 - 2 = 1$

Now do these. Some are 'add,' and some are 'take away.'

(1)	$6 + 3 =$	....
(2)	$8 - 5 =$	....
(3)	$6 - 2 =$	....
(4)	$9 - 3 =$	....
(5)	$8 + 21 =$	....
(6)	$6 - 4 =$	....
(7)	$10 - 3 =$	....
(8)	$27 + 9 =$	....
(9)	$14 - 6 =$	....
(10)	$19 - 5 =$	....
(11)	$17 - 8 =$	....
(12)	$27 + 18 =$	....
(13)	$26 - 14 =$	....
(14)	$27 - 13 =$	....
(15)	$25 + 21 =$	....
(16)	$23 - 15 =$	....
(17)	$29 - 12 =$	....

Now read these, and put in the words that are left out. Like this:

I like *to* play.
I sleep in a *bed*.

Now do these. Put in just ONE word each time:

(18) Boys .... girls play ball.
(19) Each day I go .... school.
(20) Birds build .... for their eggs.
(21) Trains go faster .... trams.
(22) A rose is a kind of ....
(23) A dog has four ....
(24) A man has .... eyes.
(25) Sugar .... sweet.
(26) Birds .... in the air.
(27) I have .... toes.
(28) Fire .... the kettle boil.

Now answer these questions:

Write the answer here:

(29) What do we call a place made for a dog to sleep in? ....
(30) What do we call a place where motor-cars may be kept? ....
(31) What does a baby wear on his head and tied under his chin? ....
(32) What do we call things like ants, wasps, and flies? ....
(33) What is the tool used for driving nails into wood? ....
(34) What do we call an instrument for measuring how heavy things are? ....

(35) What do we call the openings for pennies in sweet machines? ....
(36) What do we call the woollen sheets used in a bed? ....
(37) What do we call the things on which doors hang and swing? ....
(38) What do we call the green cases in which peas and beans grow? ....
(39) What is the tool for bending wire? ....
(40) What do we call an instrument for measuring how hot anything is? ....
(41) What do we call a back kitchen where washing-up is done? ....

Look at these words:

roll, hill, hat, bill, fill, fall.

'hill,' 'bill,' 'fill,' all end in the same sound, so they are underlined.

Now look at these other sets of words. In each set find the THREE words which end in the same sound, and *underline them.*

(42) bit, day, pay, pan, don, say.
(43) hush, dish, crash, sash, sad, dash.
(44) bend, bent, pant, lent, left, sent.
(45) melt, mend, dealt, dead, health, felt.
(46) can, weld, wood, could, melt, should.
(47) seek, deck, seed, leak, been, beak.
(48) greet, late, bait, green, land, great.
(49) reed, feet, bead, beat, feed, bean.

Here is the alphabet:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Now answer these questions

Put it here

(50) What letter comes midway between G and O? ....
(51) What letter comes next but one after M? ....
(52) What letter comes next but one before T? ....
(53) What letter comes before the letter which comes before M? ....
(54) What letter comes after the letter which comes after D? ....
(55) What letter is the fifth letter after the letter which comes midway between L and R? ....
(56) What letter comes just after the fifth letter after E? ....
(57) Suppose the first and second letters of the alphabet were interchanged, also the third and fourth, the fifth and sixth, and so on. Which letter would then come 17th? ....
(58) Suppose the 3rd letter of the alphabet were crossed out, also the 6th, the 9th, the 12th, and so on. What would be the fifth letter not crossed out? ....
(59) If the letters were crossed out as in the last question, what would be the two middle letters of those left? ....

Look at these words:

long, big, round, small, large, black.

'big' and 'large' mean the same, and 'small' means just the opposite of the other two, so these three words are underlined. Now in each of these other sets of words find the two words which mean the same, or nearly the same, and a third word which means the opposite of the other two, and *underline these* THREE *words.*

(60) below, begin, grow, commence, collect, end.
(61) long, lofty, love, hold, loathe, hate.
(62) full, vacant, vapid, empty, enjoy, increase.
(63) hinder, hasten, aid, advance, help, hollow.
(64) huge, tall, soft, tiny, vast, vacant.
(65) pour, drain, empty, draw, flow, fill.
(66) probable, shallow, profound, cloudy, clear, deep.
(67) swollen, increase, create, lessen, least, diminish.
(68) crowd, concur, cross, dissent, agree, desire.
(69) attempt, audacity, timidity, lucidity, brazen, boldness.
(70) shrink, shout, expend, contract, contest, expand.

Now in each of the following sets of numbers you have to *give the two numbers that come next.* Here is one answered for you:

2, 4, 6, 8, 10, (12), (14).

Here 12 and 14 are the numbers to be written in the brackets because they ought to come next. Now do these in the same way, putting in the brackets the two numbers that come next.

(71) 3, 4, 6, 9, 13, ( ), ( ).
(72) 1, 2, 4, 8, 16, ( ), ( ).
(73) 6, 7, 10, 11, 14, 15, ( ), ( ).
(74) 26, 20, 15, 11, 8, ( ), ( ).
(75) 3, 2, 4, 3, 5, 4, 6, ( ), ( ).
(76) 20, 18, 17, 15, 14, 12, 11, ( ), ( ).
(77) 1, 3, 5, 6, 8, 10, 11, 13, ( ), ( ).
(78) 1, 4, 9, 16, 25, ( ), ( ).
(79) 2, 3, 6, 7, 14, 15, 30, ( ), ( ).

Look at this:

snow, white—(rain, grass, blade, green, wood).

Snow is white and grass is green, so 'grass' and 'green' are underlined.

Now see this:

hat, head—(face, hand, foot, glove, mouth, shoe).

A hat is worn on the head and a shoe on the foot, so 'shoe' and 'foot' are underlined.

Do these others in the same way. *Underline just* TWO *words each time:*

(80) sheep, wool—(coat, feathers, wear, egg, bird, tree, field).
(81) eye, see—(leg, run, ear, say, man, hear, boy).
(82) foot, hand—(walk, head, run, leg, skin, road, arm).
(83) water, air—(rest, bird, tree, sea, branch, fish, swim).

(84) worse, bad—(ill, best, worse, better, sick, evil, good).
(85) rare, common—(some, all, many, never, always, few, often).
(86) ruler, clock—(hour, face, when, time, hands, strike, length).

A man selling apples and pears offers his apples at 4*d.* a pound and his pears at 6*d.* a pound. I buy an EXACT NUMBER of pounds of fruit and I pay with a SINGLE COIN, receiving no change. *Now answer these questions:*

(87) What is the greatest weight of fruit I can buy for 2*s.*? ....
(88) What is the greatest weight of fruit I can buy for 6*d.*? ....
(89) What is the smallest coin I can spend on equal weights of pears and apples? ....
(90) What is the smallest coin I can spend on apples? ....
(91) What is the smallest coin I can spend if I buy half as many pounds of apples as I do of pears? ....

Now try this:

Tom is five years old and Jack is nine. Answer these questions:

(92) How old was Jack when he was twice as old as Tom? ....
(93) How old was Tom when Jack was three times as old as he? ....
(94) In how many years will their ages added make 24? ....
(95) In how many years' time will Tom be twice as old as Jack was when Tom was three? ....

A number of small wooden cubes, each with an edge of one inch, are stuck together to form a solid cube of edge 4 inches. The big cube is then marked on each face by straight lines forming a cross, the lines being drawn from corner to corner. *Answer these questions:*

(96) How many little cubes have no face exposed on the surface of the big cube? ....
(97) How many little cubes have faces exposed but are not marked by a line drawn across a face? ....
(98) How many little cubes are marked on 1 face only? ....
(99) How many little cubes are marked on 2 faces only? ....
(100) How many little cubes are marked on 3 faces only? ....

Now that you have finished the tests (not before), if you have not yet been told to stop, write below a short account of what you think of this examination.

APPENDIX II

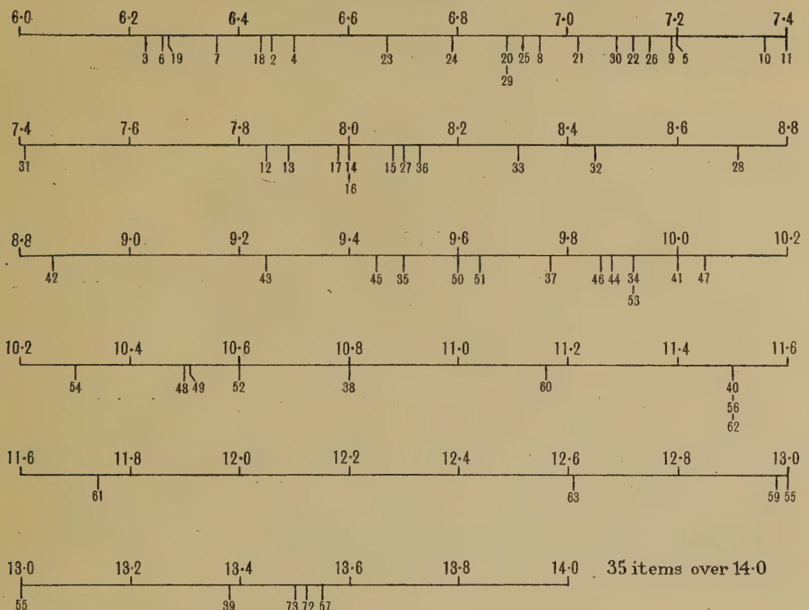
PAR AGES AND SCALE VALUES OF '*K*' TEST ITEMS

Item No.	Par age	Scale value	Item No.	Par age	Scale value
1	Below 6·0	−1·33	51	9·64	0·96
2	6·46	−0·80	52	10·6	1·21
3	6·23	−0·91	53	9·92	0·99
4	6·5	−0·77	54	10·3	1·01
5	7·2	−0·23	55	13·0	2·06
6	6·26	−0·89	56	11·5	1·59
7	6·36	−0·85	57	13·6	2·45
8	6·95	−0·44	58	12·98	2·05
9	7·19	−0·25	59	13·65	2·47
10	7·36	−0·14	60	11·16	1·46
11	7·4	−0·08	61	11·74	1·53
12	7·85	0·13	62	11·5	1·59
13	7·89	0·15	63	12·61	2·04
14	8·0	0·21	64	Over 14	2·84
15	8·08	0·27	65	”	2·41
16	8·0	0·205	66	”	3·07
17	7·98	0·20	67	”	2·895
18	6·44	−0·81	68	”	2·34
19	6·27	−0·94	69	”	2·33
20	6·89	−0·47	70	”	2·36
21	7·02	−0·37	71	”	2·44
22	7·12	−0·29	72	13·6	2·23
23	6·67	−0·66	73	13·55	2·21
24	6·79	−0·57	74	Over 14	2·44
25	6·92	−0·46	75	”	2·53
26	7·15	−0·27	76	”	2·36
27	8·1	0·23	77	”	2·63
28	8·71	0·60	78	”	3·02
29	6·89	−0·51	79	”	2·95
30	7·09	−0·31	80	”	2·21
31	7·41	−0·04	81	”	2·62
32	8·45	0·51	82	”	3·07
33	8·31	0·35	83	”	3·66
34	9·92	1·005	84	”	3·92
35	9·5	0·91	85	”	3·23
36	8·13	0·29	86	”	3·66
37	9·77	0·92	87	”	2·34
38	10·8	1·31	88	”	2·53
39	13·38	2·15	89	”	3·43
40	11·5	1·59	90	”	3·45
41	10·0	1·02	91	”	4·47
42	8·86	0·56	92	”	2·85
43	9·25	0·82	93	”	3·14
44	9·88	1·00	94	”	3·065
45	9·45	0·92	95	”	3·36
46	9·86	0·91	96	”	4·18
47	10·05	1·04	97	”	4·75
48	10·5	1·17	98	”	4·45
49	10·51	1·18	99	”	4·36
50	9·6	0·91	100	”	4·12

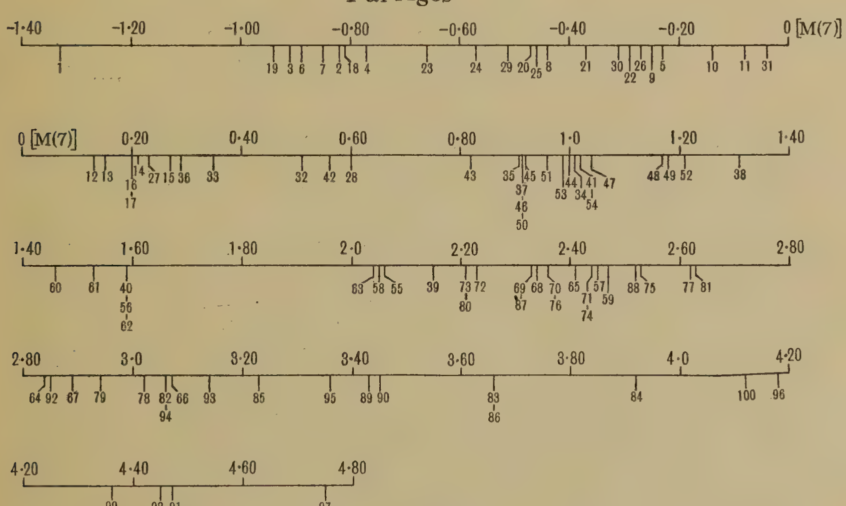
Note. The scale values from No. 64 onwards are approximate only.

APPENDIX II

IN DIAGRAMMATIC FORM



Par Ages



Scale Values

APPENDIX III

PROOF OF FORMULAE (A) AND (B)

Assume that the distribution of intelligence in two age-groups, p and q, can be represented, as in the diagram below, on one and the same base line.

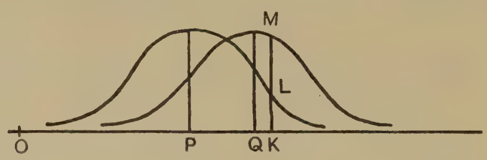


Fig. 1

Let the mean of the intelligence of the p group, measured from an arbitrarily selected origin O, be $M\ (p)$ and its standard deviation $\sigma\ (p)$. Let $M\ (q)$, $\sigma\ (q)$ be the corresponding values for the q group. Then

$$OP = M\ (p);\ OQ = M\ (q).$$

Consider any given test item. Suppose $P\ (p)$ per cent. of the individuals in the p group, and $P\ (q)$ per cent. of the individuals in the q group, correctly answer this item. Erect an ordinate KL to the distribution curve of the p group such that it cuts off an area equal to $P\ (p)$ per cent. of the total area under the curve. Then if KL produced meets the q distribution curve at M, the ordinate KM to that curve will cut off an area equal to $P\ (q)$ per cent. of the total area under the curve.

Suppose the distance PK is $X\ (p)$ times $\sigma\ (p)$, and the distance QK is $X\ (q)$ times $\sigma\ (q)$.

Then $$OP + PK = OK = OQ + QK,$$

i.e. $$M\ (p) + X\ (p)\,\sigma\ (p) = M\ (q) + X\ (q)\,\sigma\ (q),$$

or $$X\ (p) = \frac{M\ (q) - M\ (p)}{\sigma\ (p)} + X\ (q) . \frac{\sigma\ (q)}{\sigma\ (p)} \qquad \text{......(1)}.$$

But we can arrive at the relation between $X\ (p)$ and $X\ (q)$ by another method of approach. For we have seen that, granted our initial assumption as to the possibility of representing the distribution of intelligence of different age-groups on the same base line, we can deduce the percentage success with a given item in one group if we know the percentage success with the same item in another group. That is $P\ (p)$ and $P\ (q)$ uniquely determine one another, each being defined by the position of the point K, which is in a fixed position for a given item. Hence $X\ (p)$ and $X\ (q)$ also uniquely determine one another, so that the series of $X\ (p)$ values for the various test items must correlate perfectly ($r = 1$) with the series of $X\ (q)$ values for the same items. This is the statistical criterion, referred to in the main text, of the applicability of Thurstone's method. Now if two sets of measures, x and y, correlate perfectly, each x will be equal to the corresponding y, *provided that both are expressed as deviations from their respective means and in terms of their respective standard deviations as units.*

Let $m\ (p)$ be the mean, and $S\ (p)$ the standard deviation of the $X\ (p)$ values for the n 'overlapping' test items for the two age-groups, i.e. the test items correctly answered *by some, but not all,* of each group taken separately.

Then $m(p) = \frac{\Sigma X(p)}{n}$, $S(p) = \sqrt{\frac{\Sigma X^2(p)}{n} - m^2(p)}$*.

Similarly for the $X(q)$ values

$$m(q) = \frac{\Sigma X(q)}{n}, \quad S(q) = \sqrt{\frac{\Sigma X^2(q)}{n} - m^2(q)}.$$

Now, since the $X(p)$'s correlate perfectly with the $X(q)$'s,

$$\frac{X(p) - m(p)}{S(p)} = \frac{X(q) - M(q)}{S(q)}$$

as explained above. That is,

$$X(p) = \left[m(p) - \frac{S(p)}{S(q)} m(q)\right] + \frac{S(p)}{S(q)} X(q) \qquad \text{......(2).}$$

Equation (2) must be identical with equation (1), for each expresses the relation between $X(p)$ and $X(q)$.

Equating corresponding terms in the two equations, we have

$$\frac{\sigma(q)}{\sigma(p)} = \frac{S(p)}{S(q)} \qquad \text{......(A),}$$

and $$\frac{M(q) - M(p)}{\sigma(p)} = \left[m(p) - \frac{S(p)}{S(q)} m(q)\right] = \left[m(p) - \frac{\sigma(q)}{\sigma(p)} m(q)\right]$$

or $$M(q) = \sigma(p)\, m(p) - \sigma(q)\, m(q) + M(p) \qquad \text{......(B).}$$

* For $S(p) = \sqrt{\frac{\Sigma[X(p) - m(p)]^2}{n}} = \sqrt{\frac{\Sigma X^2(p) - 2m(p)\,\Sigma X(p) + nm^2(p)}{n}}$

$$= \sqrt{\frac{\Sigma X^2(p) - 2nm^2(p) + nm^2(p)}{n}} = \sqrt{\frac{\Sigma X^2(p)}{n} - m^2(p)}.$$

PART II

FURTHER ANALYSIS OF THURSTONE'S METHOD AND OF THE GROWTH CURVE

By C. A. RICHARDSON and C. W. STOKES

The first part of this monograph was devoted to an account of the application of Thurstone's method to the data obtained from the results of a group intelligence test set to a complete population. Thurstone assumes that the distributions of the intelligence of different age-groups can be represented on the same base line, and gives a criterion for determining whether this condition does, in fact, hold. In the present investigation the criterion was found to be well satisfied. But there is another assumption implicit in Thurstone's method, namely that the ability tested by each separate test item is the same—Thurstone calls it 'test intelligence.' An examination of his procedure (described in Part I) makes it clear that, unless this assumption is at least approximately true, the whole method fails. One of our present purposes, therefore, is to determine, by an analysis of the data given by each test item, how far the assumption is true.

It is also intended to inquire in greater detail into the nature of the ability growth curve indicated by the results obtained in the case of each item of the scale. Fortunately there is here a clue to direct our inquiry. Dr S. A. Courtis, of the University of Michigan, has obtained a large amount of evidence indicating that natural growth processes in various fields, given reasonably constant conditions of 'nurture,' follow a course described by Gompertz' equation $Y = Ca^{b^x}$ (see e.g. *Factors Conditioning Growth*, by S. A. Courtis). In this equation Y is the amount of growth at time x, C is the amount of growth at maturity (i.e. it is the maximum value of Y and is clearly approached asymptotically), b is a constant depending on the particular process and the particular individual, and a is a constant depending on these factors and also on the origin from which the time x is measured. a and b are both less than 1. If we put $y = Y/C$, so that y is the *proportional* growth at time x, the equation becomes $y = a^{b^x}$, a more useful form in general. Any curve given by an equation of this form is to be called a G curve.

An attempt will be made to discover whether the ability growth curves obtained from the results of the various test items are G curves or approximations to such; and, if so, how close is the approximation.

At this point attention should be called to an important distinction, namely, that between the percentage of children of a given age-group who pass a certain test and the mean percentage development in the age-group in question of the kind of ability required to pass the test. The *amount* of ability required to pass the test will be the mean ability of the age-group, 50 per cent. of whom pass the test. Thus, for example, consider the test item 63. Let A be the kind of ability required to pass this test. Now almost exactly 50 per cent. of the 12 + age-group pass this test, and it is found by a method shortly to be described that the mean value of the A ability of this group is 70 per cent. of the mean maturity value of A ability, that is of the maturity value in the case of the 'average' individual. In the 7 +

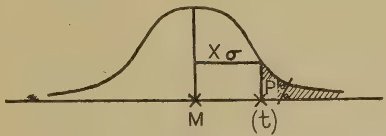


Fig. 1

age-group, A ability is found to be 49 per cent. developed, but only 0·84 per cent. of this group pass item 63. This means that, although the group reaches on an average a development of 49 per cent. in the *kind* of ability required to pass the test, only 0·84 per cent. of this group reach the *degree* of development of the ability required to pass the test, namely 70 per cent.

It must now be inquired whether, given the percentage (P) of a group passing the test, the mean degree of development (M) of the group in the kind of ability required can be determined. Let t be the amount of ability required to pass the test, so that t is the value of M for the group, half of whom pass the test. Fig. 1 above represents the distribution of the ability of the group, P per cent. of whom pass the test, the base line of the distribution being the scale of test ability.

It is evident from Fig. 1 that we have a relation $t = M + X\sigma$, where σ is the standard deviation of the distribution and X is determined solely by P, the percentage passing the test. Clearly, for a given test, $M + X\sigma$ is a constant independent of age, being always equal to t. But, equally clearly, M cannot be determined from P

alone, even in terms of t, owing to the presence of the second unknown σ. Hence a further assumption must be made, and the most likely one, and the one actually adopted, is that $\sigma/M = k$, a constant. This was indicated by the analysis described in Part I and it will also be assumed that k is the same for each test item and equal to 0·18, the value obtained in Part I. M is of course measured from 'absolute zero,' that is, from the mean of the group for which $\sigma = 0$. Our inquiry therefore reduces to this: Assuming that the coefficient of variation, σ/M, is constant and the same for each item, and is equal to 0·18, how far do the ability growth curves conform to G curves, and how far is Thurstone's implicit assumption satisfied, namely, that the same ability is involved in each item, or, in effect, that the growth curve given by each item is the same?

More than one method of tackling this question exists. The first method adopted may best be illustrated by the following typical set of calculations. They are for item 71, and it should be remembered that if $M(x)$ is the mean ability at age x and $\sigma(x)$ the standard deviation, then, since t_{71} (the ability required to pass the test) is equal to $M(x) + X\sigma(x)$ and since we assume

$$\frac{\sigma(x)}{M(x)} = 0{\cdot}18, \text{ it follows that } M(x) = \frac{t_{71}}{1 + 0{\cdot}18X};$$

or, if we choose t_{71} as our unit for this item,

$$M(x) = \frac{1}{1 + 0{\cdot}18X}.$$

Table I

Item 71

1	2	3	4*	5	6	7
Age	X	$0{\cdot}18X$	$\frac{1}{1+0{\cdot}18X}$ (M)	M as percentage of C the maturity value	Expected value of M/C as a percentage	Percentage discrepancy
6+	—	—	—	—	—	—
7+	+2·66	0·48	0·6757	63	63	0
8+	+1·89	0·34	0·7463	70	70	0
9+	+1·41	0·254	0·7974	75	76	−1·3
10+	+0·92	0·166	0·8576	80	80	0
11+	+0·575	0·104	0·9061	85	84	+1·2
12+	+0·46	0·083	0·9232	86	87·5	−1·8
13+	+0·17	0·031	0·9700	90	90	0

* Estimated maturity value $C = 1{\cdot}07$.

From the data in column (4), the maturity value C, of M, is estimated *on the assumption that M is following a G curve.* This estimate was made by inspection of the graph of $\log M$, and is probably accurate to within about 5 per cent., rather than by the theoretically more accurate algebraic method, which is too sensitive in this case to slight irregularities in individual data. The percentages in column (5) are then computed, and tested for fit to a G curve. If

$$\frac{M}{C} = a^{b^x}, \text{ then } \log\log\frac{M}{C} = x\log b + \log\log a,$$

i.e. the relation between time (age) x, and $\log\log(M/C)$ is linear. Hence the quickest method of finding whether a set of percentages follow a G curve is to plot them on graph paper one dimension of which is on a log log scale*. On this paper the plot will be a straight line if the percentages follow a G curve. Column (5) was actually tested in this way, and from the result it was possible to determine the value of the percentages which were to be expected if they followed a G curve exactly. Column (7) gives the percentage discrepancy between the actual and the 'expected' values. Evidently the net result of the calculations is to show that, for item 71, the values of M given in column (4) do in fact lie very close to a G curve having a maturity value for M of 1·07.

The next point is that, since the percentages given in column (5) are independent of the particular unit employed for the item, this list of percentages *ought to be the same for every item*, if Thurstone's assumption is correct, for then we should obtain the same growth curve for each item, since the ability is of just the same kind in each case. Hence the set of calculations, of the type given, for the hundred items of the test, provide a means of deciding, not only whether the growth curves are G curves, but also whether the ability tested by each item is effectively the same.

The percentages corresponding to those given in column 5 of Table I were computed for 77 items of the scale, and are given in detail in Appendix I to this part. It will be seen that the values for similar test items are more alike than those for different types of test items. In the case of the remaining 23 items, the data were either insufficient or somewhat too irregular for it to be possible to estimate the maturity values to a reasonably close approximation. The results for the 77 items referred to may be summarized as follows:

* I am indebted to the kindness of Dr Courtis for a supply of this graph paper. C. A. R.

Table II

Age	Mean percentage development	Mean variation ignoring sign
6+	56	2·8
7+	63	3·1
8+	69	2·9
9+	74	2·7
10+	78	2·5
11+	83	2·0
12+	87	1·8
13+	90	1·4

The second column in this table contains the mean, at each age, of the percentage development obtained from the various items, while the third column gives the mean variation about this mean. This averaging is justified because the variations of the maturity values for the different items, when expressed in terms of the same unit, are relatively small. If Table II be considered in conjunction with the detailed results given in Appendix I it will be seen that, in spite of the occurrence of occasional abnormal values (especially in the earlier years), the growth curves given by the different items do, in fact, cluster fairly closely about a mean curve, and, to this extent, Thurstone's assumption is justified, though it can be regarded only as a first-order approximation. The comparison of the mean curve with the G curve of nearest fit is given in Table III, where the third column gives the value of percentage development to be expected if the curve is an exact G curve, and the fourth column gives the percentage discrepancy between the actual and the expected values.

Table III

Age	Actual value	Expected value	Percentage discrepancy
6+	56	56	0
7+	63	63	0
8+	69	69	0
9+	74	74·5	−0·7
10+	78	79	−1·3
11+	83	83	0
12+	87	86	+1·2
13+	90	89	+1·1

It is therefore apparent that the mean growth curve obtained is almost exactly a G curve. The closeness with which the growth curves for the separate items approach to G curves is clearly shown later in Fig. 3, which is obtained by a method shortly to be described.

We may then summarize our results so far as follows: On the assumption that the coefficient of variation is constant for each item, and has the same value 0·18 in each case, it is found that, for the 77 items which can be dealt with by the method described, the growth curves do on the whole cluster fairly closely round a mean curve which is almost an exact *G* curve.

We may now compare the mean curve thus obtained by averaging the curves for the various items, with the curve obtained in Part I, for the scale as a whole, by Thurstone's method. The absolute values of *M* are given in Table V, Part I, and to each of these must be added 5·556 to convert it to a measurement from absolute zero. In Table IV below, the second column gives these values, the third column contains the percentage development which they express on the assumption that they follow a *G* curve, the fourth column gives the expected values for exact fit, and the fifth the percentage discrepancy.

Table IV

Age	*M* absolute	Actual percentage	Expected percentage	Percentage discrepancy
6 +	4·787	55	55	0
7 +	5·556	64	62	+3·2
8 +	6·0068	69	68·5	+0·8
9 +	6·4602	74	74	0
10 +	6·7292	77	79	−2·5
11 +	7·1508	83	83	0
12 +	7·5770	86	86	0
13 +	7·7587	89	89	0

Two facts are at once apparent from this table. In the first place we have again a nearly exact *G* curve; and, secondly, the curve is almost identical with that already obtained by the other method. It will also be noted that the 6-year-point now fits into place.

Table V

Age	Percentage development
6 +	55·5
7 +	63
8 +	69
9 +	74
10 +	78·5
11 +	83
12 +	86
13 +	89

Considering Tables III and IV together, we may take for the most probable percentage levels of *general* development at the various ages the values given in Table V.

The percentages in the second column of Table V form a practically perfect G curve, and if we follow this curve upwards and downwards we obtain by extrapolation the following values for the percentage development of the general level of intelligence at different ages:

Table VI

On attaining the age of	Percentage development
Birth	$6\frac{1}{2}$
1 year	$11\frac{1}{2}$
2 years	18
3 ,,	26
4 ,,	$34\frac{1}{2}$
5 ,,	43
6 ,,	$51\frac{1}{2}$
7 ,,	$59\frac{1}{2}$
8 ,,	66
9 ,,	72
10 ,,	$77\frac{1}{2}$
11 ,,	82
12 ,,	$85\frac{1}{2}$
13 ,,	88
15 ,,	$92\frac{1}{2}$
18 ,,	96
24 ,,	99

Although it must be emphasised that this extrapolation is, of course, speculative in character, especially from 6 years down, the curve given by these values and plotted in Fig. 2 is of considerable interest. In particular it explains the almost imperceptible increase in intelligence test norms after the age of 16 to 18 years. The earlier part of the curve shows an increasing rate of growth, and the later part a diminishing rate, as Thurstone found, but the point of inflexion comes shortly after the fourth birthday—much earlier than Thurstone's estimate. It must of course be remembered that the curve we have obtained is that for the development of the 'average' individual. The percentage of their mature development reached by particular individuals at various ages may not be the same as those found for the average individual, even though growth is following some G curve in each case. This is a point for further investigation. Moreover, abnormalities in 'nurture' or the environmental conditions may, and generally will, affect the growth curves of individuals.

We have to inquire, in fact, into two processes of averaging. The first is the averaging across the items to obtain a mean M at each age from the M's given by the different items. The second is the averaging of the ability of all the individuals in a population to obtain M, the mean ability; in other words we have to investigate the relation existing between the abilities of individuals and the mean ability curve.

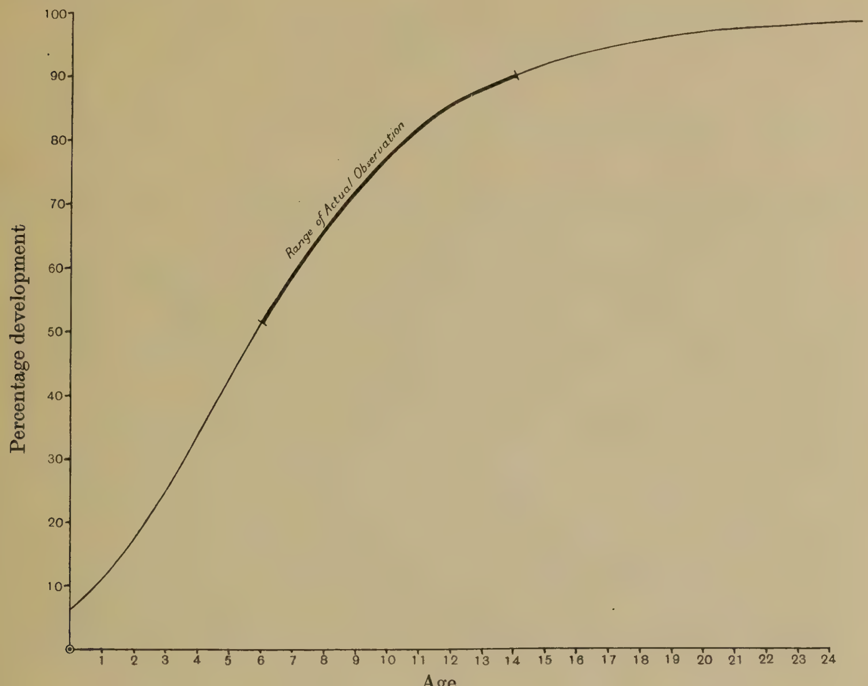


Fig. 2. Mean percentage growth curve of ability, with extrapolations.

In considering the first process we remember that for any item we have, very nearly, a mean proportional growth curve of the form

$$M = a^{b^x}.$$

Then the mean M ($\bar{M}$) for all the items is given by

$$\bar{M} = \frac{\Sigma M}{n} = \frac{\Sigma a^{b^x}}{n},$$

where n is the number of items.

Now it is clear that, for a and b varying from item to item, this is not in general a G curve. Hence it appears rather strange at first

sight that we should get a G curve for the mean curve as well as for the curves for each item. The reason is as follows:

If $$M = a^{b^x},$$
then $\log M = b^x \log a$, and $\log \log M = \log \log a + x \log b$, therefore

$$\frac{\Sigma \log \log M}{n} = \frac{\Sigma \log \log a}{n} + \frac{x \Sigma \log b}{n}.$$

Now this is the log log of the G curve

$$M_1 = a_1^{b_1^x},$$

where M_1, b_1 and a_1 are uniquely defined by

$$\log \log M_1 = \frac{\Sigma \log \log M}{n}, \quad \log b_1 = \frac{\Sigma \log b}{n}, \quad \log \log a_1 = \frac{\Sigma \log \log a}{n}.$$

Hence it is clear that the mean of the log logs of a number of proportional G curves is itself the log log of a G curve.

Therefore $\bar{M}$ will approximate to a G curve only so far as $\log \log (\Sigma M/n)$ approximates to $\Sigma \log \log M/n$. Now an inspection of the M's for the different items shows that, in fact, even for the widest differences apparent, at any particular age, the log log of the averages is very nearly equal to the average of the log logs.

This accounts for the fact that the mean proportional growth curve is very nearly a G curve, as well as the curves for the different items.

We have now to consider the relation between the abilities of individuals and the mean ability of the group they compose.

Let Y be the ability of an individual (either for a single test item or for the whole scale—it will not affect the argument) at a distance $X\sigma$ from the mean (M) of his age-group, where σ is the standard deviation of the group.

Then $$Y = M + X\sigma,$$
and putting $\sigma/M = V$, $\quad Y = M(1 + VX)$,
where V, as we have seen, is constant.

Now we have found M to be of the form Ca^{b^x}, therefore

$$Y = C(1 + VX)\, a^{b^x}.$$

If X remains constant, i.e. if the individual keeps his *relative* position in the group as he grows older, the equation for Y clearly gives a G curve. It is commonly assumed that an individual generally keeps his relative position, at any rate approximately, and it is probably true in most cases. Abnormal conditions of 'nurture,' however, using that term in its widest sense to include morbid physiological changes, may cause X to vary with age. In such a case we can perhaps best

picture the growth of the individual as following a curve approximating to a G curve, but tending to a maturity value which fluctuates with age.

Conversely if $Y_1, Y_2, \ldots$ are the abilities of the members of a group where

$$Y_1 = C(1 + VX_1)\, a^{b^x}, \quad Y_2 = C(1 + VX_2)\, a^{b^x}, \text{ etc.}$$

(note a, b and C are constant for a given item or the scale as a whole),

then
$$M = \frac{\Sigma Y}{N} = \frac{\Sigma C(1 + VX)}{N} a^{b^x}$$

(where N is the number of individuals)

$$= Ca^{b^x},$$

for $\Sigma X = 0$ at any age whether X is varying in some individuals or not.

Hence if the abilities of the members of the group follow G curves for which a and b are constant, the mean growth curve will be a G curve. The absolute maturity values for the individuals will be of the form $C(1 + VX)$ and will vary with X from one individual to another.

If the abilities for individuals followed curves quite different from G curves, or if they followed G curves with a's and b's varying widely, it is difficult to see how the mean growth curve could be a G curve except by a pure fluke. Hence the fact that we have actually found the mean growth curve to be a G curve is strong presumptive evidence that the growth curves of individuals are also G curves, with a and b constant or nearly so, but with widely varying maturity values, and therefore with wide variations in *absolute* value at any age. In the case of some individuals the maturity to which the growth tends may, as we have seen, be different at different ages. In connection with this it would be interesting to study, by means of the technique outlined in this monograph, the growth of an abnormal group, such (for example) as the defectives for whom Burt gives data in his "Mental and Scholastic Tests," to see if the mean growth curve for the group shows signs of a fluctuating maturity value, as it is often supposed that the development of defectives tends to decay prematurely. It would, of course, be necessary to determine the value of V for the group, as this might, and probably would, differ from the V for a complete population. Quite possibly it might not be constant for a selected group of that kind.

We must now return to the fact that our results so far represent an approximation only. A number of facts indicate the importance and desirability of further investigation. The variations in the growth curves given by individual test items, and the irregularities in particular growth curves, especially where these indicate certain uniformities, are alone sufficient to cause a demand for further inquiry. The best way of approaching the latter is through an analysis of our data by a method alternative to that we have so far been pursuing, which will shortly be described. But it is of interest first to analyse the properties of the G curve in greater detail.

The formula $y = Ca^{b^x}$ was chosen in the first instance on empirical grounds, and the equation in this form gives no clue as to the reason why growth processes should follow this law. It certainly gives a curve of the right form, rising asymptotically from zero and tending again to a finite limit as x increases indefinitely. There are, however, many curves of such a nature, which have been suggested from time to time for the growth curve. Nevertheless, the Gompertz curve does seem to fit the data very closely, and it is interesting to inquire whether there is any *a priori* reason suggesting the adoption of this particular law.

It would perhaps be natural, as a first crude assumption, to take the rate of growth, dy/dx, to be proportional to y, writing $dy/dx = py$, which leads to a simple exponential law $y = Ae^{px}$. This makes no provision for the decay in the rate of growth which is known to occur, and must be taken into account. The most obvious modification of the simple exponential law to allow for this is perhaps the insertion of the decay factor e^{-qx}; as is done, for instance, in modifying the Simple Harmonic Motion equation to fit the case of 'damped' oscillations. The differential equation for the growth process would then become

$$\frac{dy}{dx} = py\,e^{-qx}.$$

The solution of this equation is

$$y = Ce^{-\frac{p}{q}e^{-qx}},$$

or, writing $-p/q = \log a$, $-q = \log b$;

$$y = Ca^{b^x},$$

which is of course the equation of the G curve, and which, as has already been pointed out, is more conveniently written $y = a^{b^x}$, where y now measures the 'proportional' growth.

In this equation a, being equal to the proportional growth at $x = 0$, is a constant depending on the choice of origin—at present arbitrary. It would, from a theoretical point of view at any rate, and, in view of the procedure to be developed later, possibly also in practice, be desirable that the origin should be fixed by some intrinsic property of the curve, and that the constants employed in its equation should measure such properties. Now the obvious point on the curve to use to fix the origin is the point of inflexion; and the next step is, then, to transfer the origin so that the y axis goes through this point. Starting from the differential equation of the curve, we find

$$\frac{d^2y}{dx^2} = -pqe^{-qx}y + pe^{-qx}\frac{dy}{dx} = -pqe^{-qx}y + (pe^{-qx})^2 y$$
$$= pe^{-qx}y\,(pe^{-qx} - q).$$

Hence, if the point of inflexion is at $x = T$, it follows that

$$pe^{-qT} = q, \quad \text{i.e. } T = \frac{1}{q}\log\frac{p}{q}.$$

Changing the origin in the equation

$$y = e^{-\frac{p}{q}e^{-qx}},$$

by writing $x + T$ for x, so that e^{-qx} becomes

$$e^{-\left(qx + \log\frac{p}{q}\right)} = e^{-qx + \log\frac{q}{p}} = \frac{q}{p}e^{-qx},$$

the equation transforms into $y = e^{-e^{-\frac{x}{\theta}}}$, where θ is written for $1/q$. If now a new unit for x be taken, equal to θ of the units so far used, the equation takes finally the rather more elegant form $y = e^{-e^{-x}}$. This may be called the Standard Form of the equation of the G curve, to which any such curve may be reduced by change of origin and contraction or expansion parallel to the x axis. Any G curve is, then, specified by the two constants T and θ. The number T gives, of course, the age at which maximum growth rate occurs. Let us inquire into the nature of the property measured by θ.

The value of y at time $x + t$ is

$$e^{-e^{-x-t}} = e^{-e^{-x}.e^{-t}} = (e^{-e^{-x}})^{e^{-t}},$$

so that in any interval of age represented by t of our new units, i.e. θt years, if age was originally measured in years, y is raised to the power e^{-t} of itself. If t be taken equal to unity this power is $1/e$, so that the unit θ is the time in which y is raised to the power $1/e$ of itself, and θ is a kind of measure of the rapidity of the growth process.

The larger θ is, when expressed in years, the longer it takes to raise y to the power $1/e$ of itself, and the slower is the progress to the maturity maximum. Adapting a word coined by Dr Courtis, this unit may be called an 'isochron,' and the age in isochrons, reckoned from the age of maximum growth rate, may be called the isochronic age.

If a growth process follows this law, we know all about it if the isochron θ and the age T of maximum growth rate are known; for if, then, age is measured in isochrons from this instant, the proportional development is given by the standard curve $y = e^{-e^{-x}}$, which is the same for all processes following the law. To any isochronic age there corresponds a definite proportion of maximum development, and *vice versa*. In particular, to the isochronic age 0, there corresponds the proportional development $1/e$ (= 0·3679); or, in other words, the maximum growth rate occurs when the development is 36·79 per cent. of maturity. Differentiating $y = e^{-e^{-x}}$, and putting $x = 0$ in the result, this maximum growth rate is readily found to be $1/e$ (measured in terms of the isochron appropriate to the process, and the maturity value)*.

If a table of values of $e^{-e^{-x}}$ be constructed, it can be used to read off the percentage development corresponding to any given isochronic age; or, *vice versa*, the isochronic age corresponding to any particular percentage development. By means of this table† it is easy to test whether a process follows a G curve. It is first necessary to estimate the maturity value from the series of observed values, and to express these as percentages of this maximum. The next step is to read off from the table the isochronic ages equivalent to these percentages. If the isochronic ages so obtained are plotted against the age in years, the result should be, in the event of agreement with the G curve, a straight line—for a given increase in isochronic age should involve always the same increase in actual age. Moreover the increase in age expressed in years corresponding to unit increase in isochronic age gives the value of the θ of the process, and the age at which the line cuts the ordinary age axis is the value of T, so that the growth process under review is completely described. In order to exhibit the development in graphic form it is not necessary actually to find T and θ. It is more convenient for this purpose to use graph paper with the standard curve drawn ready on it, and to proceed as follows.

* The recurrence of this number $1/e$ is remarkable. If it be called ϵ, the standard form of the equation is $y = \epsilon^{\epsilon^{x}}$.

† See Appendix III.

The straight line drawn as described above gives the relation between isochronic age and actual age. Now, the percentage growth at certain ages, expressed in years, being known, this line may be used to convert these ages to isochronic ages, and the percentages may then be plotted against isochronic ages on the prepared graph paper, when comparison with the G curve is immediate.

The 77 items dealt with in this inquiry have all been treated in this way, and plotted against the same standard curve, with the result shown in Fig. 3. This presents in a vivid manner the degree of general agreement with this law of growth. Of course such a graph must of necessity conceal the nature of the departures of individual items from the law, and the considerable regularity in these variations that seems to persist throughout any set of similar questions. It does, however, appear to indicate a high degree of general agreement with the G curve. It must not be inferred from this graph that the growth curves for individual items are the same, for the appropriate T and θ have been used in each case to bring the curves to the standard form. The differences between the individual growth curves can readily be seen from Table VII, where the values of T and θ are given for each item.

Table VII

n	θ	T	n	θ	T	n	θ	T	n	θ	T
5	4·4	4·25	29	3·0	4·65	51	5·05	4·4	72	3·8	5·1
6	4·1	4·55	30	3·1	4·7	52	4·65	4·0	73	4·05	4·7
7	4·0	4·1	31	4·1	4·0	53	4·25	4·7	74	3·75	5·05
9	4·05	4·2	32	4·4	3·9	54	4·4	4·45	75	3·95	4·95
10	3·65	5·0	33	3·9	4·55	55	4·1	4·45	76	3·9	4·75
11	3·6	4·7	34	4·3	4·55	56	3·85	5·1	77	3·9	4·8
12	4·55	3·65	35	4·0	4·75	57	4·9	3·25	78	3·9	4·9
13	3·85	5·0	36	4·1	3·8	58	4·0	3·9	79	4·4	4·2
14	4·0	4·9	38	4·7	3·35	60	4·05	5·2	80	4·2	4·7
15	3·95	4·4	39	5·25	1·35	61	3·85	5·3	81	4·4	4·0
16	3·5	4·9	40	4·2	4·55	62	4·25	5·4	82	4·4	3·8
17	4·05	4·9	41	4·35	3·7	63	4·9	4·7	83	4·6	3·4
18	7·2	2·0	42	3·8	5·1	65	4·4	4·3	84	5·55	1·55
20	6·05	3·95	43	4·35	4·3	66	4·35	3·85	85	4·8	3·1
21	3·1	4·65	44	4·8	4·4	67	4·35	4·55	86	5·05	3·0
22	4·5	1·65	46	3·95	4·7	68	4·3	4·05	87	3·9	5·15
23	3·2	4·55	47	4·1	4·5	69	4·8	3·3	88	4·05	3·8
24	3·6	4·05	48	4·25	4·4	70	3·75	5·45	89	3·6	5·1
26	4·35	3·2	50	4·8	4·5	71	4·1	4·4	90	4·7	3·5
27	4·4	2·35									

The values for the means given in Table II are $\theta = 4{\cdot}35$, $T = 4{\cdot}15$, and those for the individual items will be seen to cluster fairly closely round these. In order to show the differences in the G curves produced

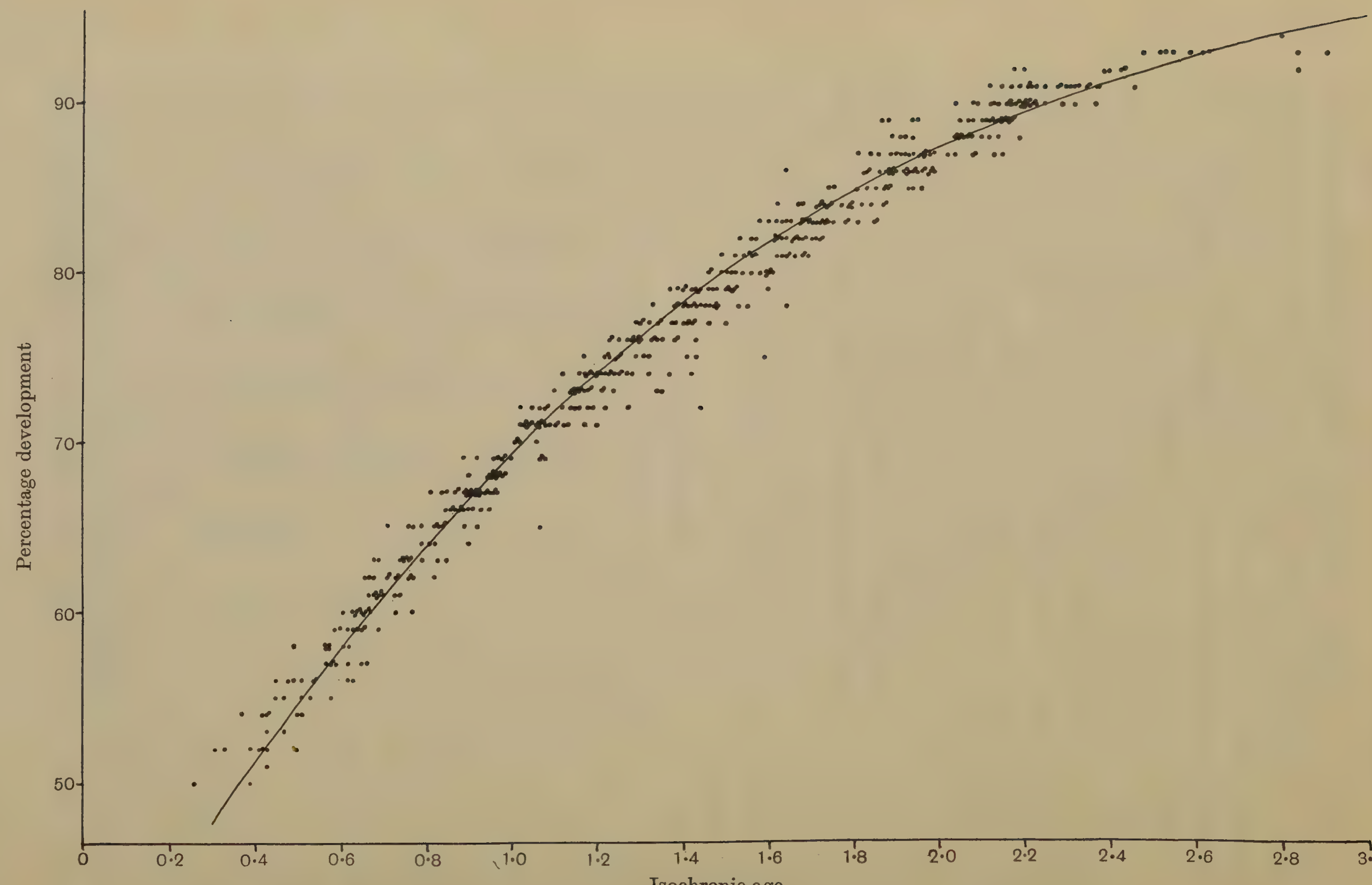


Fig. 3. Comparison of the plots yielded by the individual items with the standard G curve.

by variations in the values of θ and T of the order of those occurring in Table VII, the curves for two extreme cases, and for the means, have been plotted on the same scale. Broadly speaking, a high value of θ seems to correspond to a low value of T. Bearing this in mind, the items chosen for this comparison were No. 84, which has a comparatively high θ and low T, and No. 23, in which the reverse is the

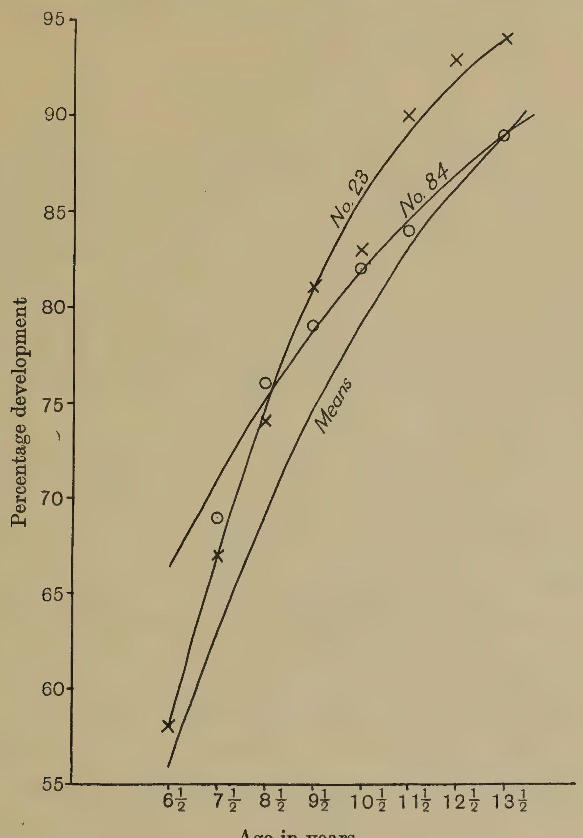


Fig. 4. Comparison of the growth curves for items 23 and 84 with the mean curve.

case. While examining these curves, which are shown in Fig. 4, it must be borne in mind that the curves have been deliberately chosen as of opposite types, and that the divergence will not, in general, be anything like so much as that exhibited by these two. Extreme types such as these are not of frequent occurrence in Table VII, in fact in only six cases among the 77 items is θ greater than 5, and it would

seem fair to say that the individual curves do cluster fairly closely round the mean, at any rate over the observed range.

Proceeding now to a more detailed investigation of Thurstone's method, and of the results obtained from it, it is convenient here to recapitulate the mathematical basis. The general problem presented by the tables giving the percentages of children at the various ages who answer correctly the hundred items of the K test, may be stated in the form of the set of equations

$$X_n = \frac{t_n - M_n}{\sigma_n} \qquad \text{......(1).}$$

Here n is the number of the item. It is assumed that the ability of a child with respect to the item n can be measured on some scale; and that this ability is, at any given age, distributed normally among the population. M_n is the mean value of this ability, and σ_n the standard deviation; both being functions of the age, x. t_n is the amount of ability needed to answer the question correctly, and is independent of x. X_n is the equivalent normal deviation corresponding to the percentage of children who answer the question correctly, and is, of course, a function of x.

It is quite clear that the problem of fitting functions M_n and σ_n, and constants t_n to these equations, in which only the functions X_n are known, is hopelessly indeterminate, and that solutions exist in infinite variety. We may in fact take M_n or σ_n to be any function of x that we please, or alternatively assume any relation between M_n and σ_n. The constants t_n will still be at our disposal. In attempting to arrive at the explanation of the results of any experiment, however, the scientist chooses the simplest hypothesis that will fit the facts. One of the simplest possible theories in the present case is that the same ability function may do for every item, i.e. that, for all practical purposes, the questions all test the same ability. M and σ, dropping the suffixes, become functions of x alone, and our set of equations is now

$$X_n = \frac{t_n - M}{\sigma} \qquad \text{......(2).}$$

It is this assumption that is made in Thurstone's method. There is no harm in making such a tentative assumption, indeed it is only in this way that progress can be made, but it is important to consider how far, in the sequel, the results justify the assumption, and how far deviations from it invalidate the analysis, or vitiate the results.

It is the object of this inquiry to see how closely a common M function can be fitted to the given data, and to discuss the deviations from this function.

It is easy to find the condition that has to be satisfied by the given X_n if a common M function is to be a possible solution, in fact it follows at once from equations (2) that we must have

$$\frac{X_2 - X_1}{t_2 - t_1} = \frac{X_3 - X_1}{t_3 - t_1} = \frac{X_4 - X_1}{t_4 - t_1} = \text{etc.} \qquad \ldots\ldots(3),$$

or in other words, the ratios $X_2 - X_1 : X_3 - X_1 : X_4 - X_1$: etc., must be independent of x, the age. Now it is not possible to apply this condition directly, for comparatively small variations in the values of the X_n will lead to wide variations in the values of the ratios of these small differences. It is possible, however, to obtain useful qualitative criteria on these lines. Suppose that the values of X for several items be plotted against the age, on the same axes. If these curves are irregular in their behaviour, that is if some of them seem to diverge from one another, while others converge, and others again cut right across, then it would seem very improbable that a common M function could be found to fit the items concerned. If, on the other hand, the curves on the whole seem to converge more or less steadily, or to remain roughly parallel, or to diverge fairly evenly, then it would look as though a common M function might exist for these items. What is more, parallel X curves would imply a constant σ, converging ones would mean σ increasing, and diverging ones σ decreasing. This follows at once from the equation

$$X_p - X_q = \frac{t_p - t_q}{\sigma},$$

which is an immediate deduction from equations (2), p and q being the numbers of any two items.

The curves* obtained from the values of X for the items 42 to 49 are given in Fig. 5, and those for items 60 to 70 in Fig. 6. The behaviour of the first set of curves does not suggest any too strongly the existence of a common M function. There is some cutting across, and no great uniformity in the matter of convergence or divergence. They do however follow similar lines, and, while we should not expect to be able to find accurately a common M function, we may be able to find one that will serve very approximately.

* When these X curves are plotted, a certain amount of irregularity is exhibited when X is greater than 2. As this is certainly in part due to the larger probable error existing in these cases, it has been thought fit, throughout this section of the inquiry, to exclude such values altogether.

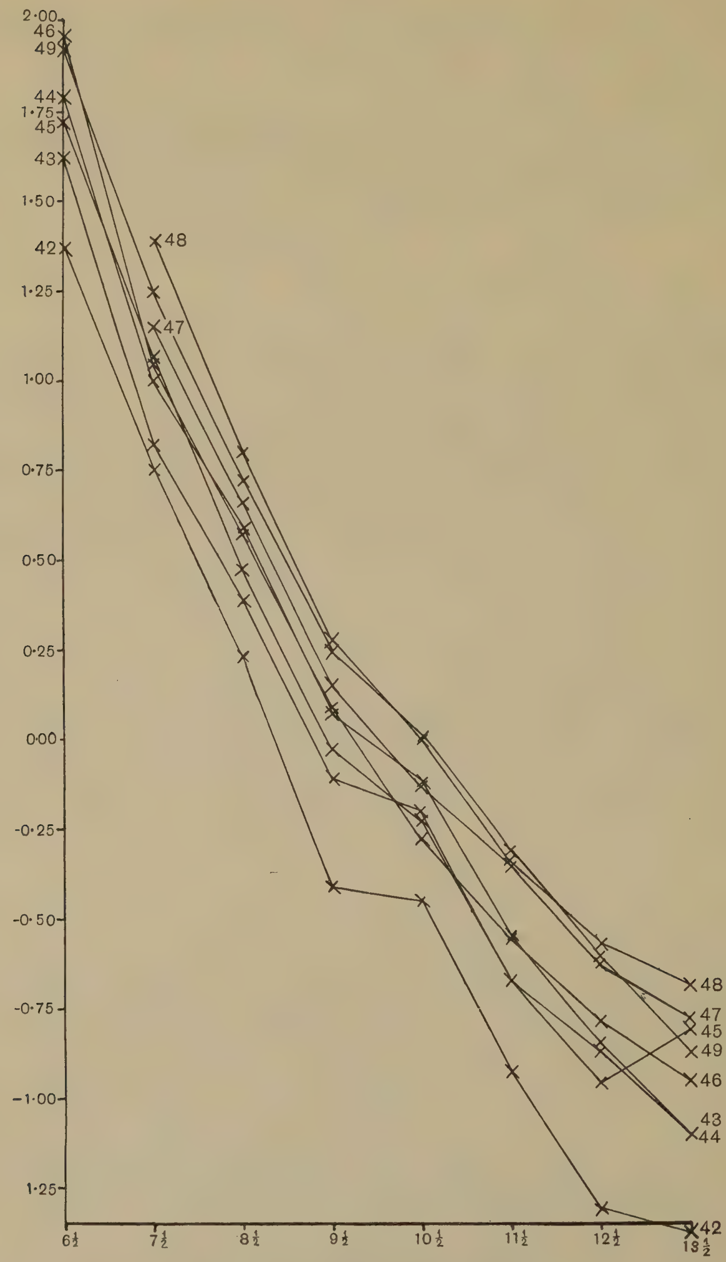


Fig. 5. X curves for items 42 to 49.

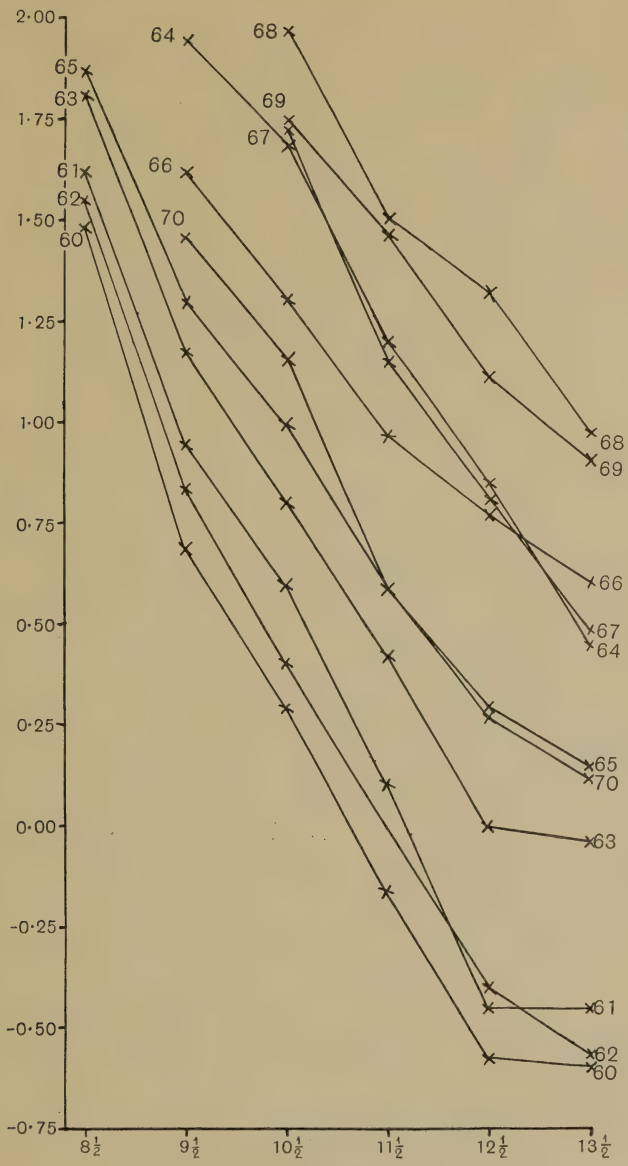


Fig. 6. *X* curves for items 60 to 70.

In the second set, apart from irregularity in the behaviour of 69 at age 12, and of 70 at ages 9 and 10, and excepting altogether items 64 and 67, whose graphs cut distinctly across the others, there seems to be a fair indication of the possibility of a common *M*. The set of

questions 60, 61, 62, 63, 65 and 66 give, however, a distinctly diverging set of curves. This means a decreasing σ, which is not easy to imagine. On the other hand, the three questions 66, 68 and 69 give, if anything, converging graphs, indicating an increasing σ.

When we come to examine the curves for items taken from different sets of questions, it becomes difficult to imagine how even an approximate common M function can exist. In Fig. 7 are drawn the graphs for questions 40, 57, 61, 71 and 88. A mere glance suffices to convince us that the condition (3) is nowhere near satisfied.

Grave doubts might then be entertained as to the possibility of fitting a common M function to the whole set of questions, sufficient at any rate to make a thorough investigation of the whole matter essential. Let us see in the first place to what extent this hypothesis of a common M is used in Thurstone's method. The critical point in the proof, given in Part I, Appendix III, is the equating of coefficients between the two equations numbered (1) and (2) there. If the $M\ (p)$ and $M\ (q)$, $\sigma\ (p)$ and $\sigma\ (q)$ vary from item to item, the first of these ceases to be an equation between variables $X\ (p)$ and $X\ (q)$. It is then simply a typical one of a set of equations, the coefficients in which vary from equation to equation. This equating of coefficients is presumably inadmissible unless the variation in the coefficients $\{M(q) - M(p)\}/\sigma(p)$ and $\sigma(q)/\sigma(p)$ from item to item is small compared with the magnitude of these quantities. The only way to satisfy ourselves on this point is to find, if possible, the closest possible common M and σ functions that will fit the data. This must to a certain extent be a matter of trial and error, but, without some guiding principles, it would be an impossible task, considering the hundreds of equations involved. Before proceeding further then, it will be convenient to give a little analysis that will be used here and later in the inquiry.

If the condition (3) of page 41 is satisfied, then we know the ratios $t_2 - t_1 : t_3 - t_1 : t_4 - t_1 :$ etc., and can therefore find any of the t_n in terms of $t_2 - t_1$ and t_1. In fact we have a set of equations $t_2 = t_1 + A$, $t_3 = t_1 + Ad_3$, $t_4 = t_1 + Ad_4$, etc., in which the d's are known, and A is written for $t_2 - t_1$.

We then have, from equations (2),

$$\frac{t_2 - t_1}{\sigma} = X_2 - X_1,$$

giving $$\sigma = A/(X_2 - X_1),$$

and finally $$M = t_1 - \sigma X_1 = t_1 - \frac{AX_1}{X_2 - X_1},$$

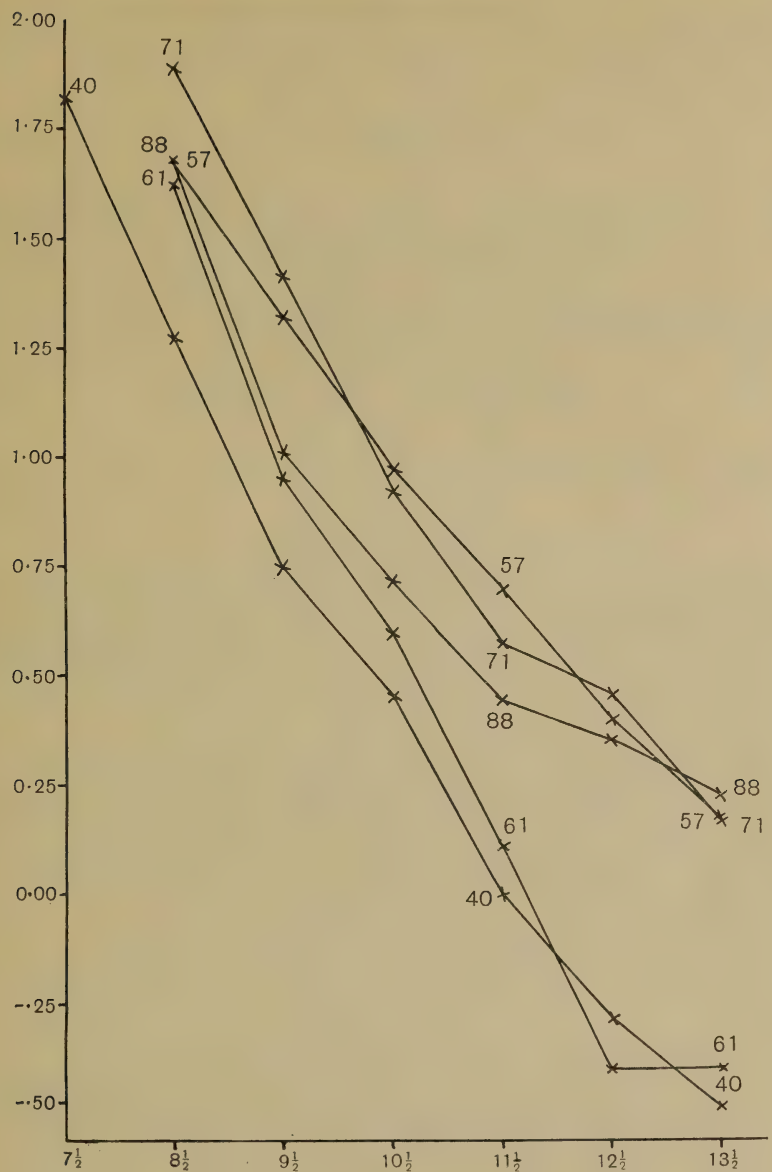


Fig. 7. X curves for items 40, 57, 61, 71, 88.

so that M and σ are known as functions of the age x. At first sight it would appear that this solution involves two arbitrary constants, but this is probably largely illusory, for in practice it would be

necessary to define a scale of measurement of M, involving the fixing of a unit and a zero. If A be taken as our unit and t_1 as the origin from which M is measured, the solution takes the form

$$\sigma = \frac{1}{X_2 - X_1}, \quad M = -\frac{X_1}{X_2 - X_1} \qquad \text{......(4).}$$

This is not put forward as a practical method of finding M and σ, for reasons already stated, but it seems to be sound theoretically, and it leads to a useful practical method of investigation. Before developing this, however, it may be helpful to give an illustrative example of the analysis as so far explained.

Let us take a set of five hypothetical items, in which the X curves are

$$X_1 = -\tfrac{1}{7}(x - 3),$$
$$X_2 = -\tfrac{1}{6}(x - 4),$$
$$X_3 = -\tfrac{1}{5}(x - 5),$$
$$X_4 = -\tfrac{1}{4}(x - 6),$$
and
$$X_5 = -\tfrac{1}{3}(x - 7),$$

i.e. a set of straight lines through the point (10, − 1), forming a very plausible group. The condition for common M and σ functions is clearly satisfied, and, by considering the intercepts on the X axis, we find

$$\frac{X_2 - X_1}{5/21} = \frac{X_3 - X_1}{4/7} = \frac{X_4 - X_1}{15/14} = \frac{X_5 - X_1}{40/21}.$$

Replacing the $X_2 - X_1$, etc., by $t_2 - t_1$, etc., and multiplying the denominators by 42,

$$\frac{t_2 - t_1}{10} = \frac{t_3 - t_1}{24} = \frac{t_4 - t_1}{45} = \frac{t_5 - t_1}{80} \text{ (} = A \text{ say),}$$

and thus

$$t_2 = t_1 + 10A,$$
$$t_3 = t_1 + 24A,$$
$$t_4 = t_1 + 45A,$$
$$t_5 = t_1 + 80A,$$

whence
$$-\tfrac{1}{7}(x - 3) = X_1 = \frac{t_1 - M}{\sigma},$$

and
$$-\tfrac{1}{6}(x - 4) = X_2 = \frac{t_1 + 10A - M}{\sigma}.$$

Subtracting these last two equations we find

$$\frac{10A}{\sigma} = -\frac{x}{42} + \frac{5}{21},$$

i.e.

$$\sigma = \frac{420A}{10 - x},$$

and therefore

$$M = t_1 + \tfrac{1}{7}(x - 3)\,\sigma = t_1 - \frac{60A\,(x - 3)}{x - 10}.$$

If t_1 be taken as our zero, and A as our unit of measurement for M, this becomes

$$\sigma = \frac{420}{10 - x} \quad \text{and} \quad M = \frac{60\,(x - 3)}{10 - x}.$$

It is to be noticed that there is a linear relation between M and σ, namely

$$M = \sigma - 60.$$

If, after Thurstone, a new, absolute, zero for M be defined as the value of M in the present scale at which σ is zero, then this relation takes the form

$$\frac{\sigma}{M} = 1.$$

The new zero is at $M = -60$ on the old scale, while the unit measure of ability is still A. The values of t_1, t_2, t_3, t_4, t_5 are now 60, 70, 84, 105 and 140. It is readily seen that with these values for the t_n, and with

$$M = \sigma = \frac{420}{10 - x},$$

the equations

$$X_n = \frac{t_n - M}{\sigma}$$

are satisfied.

To return to the general case, the solution given in equations (4) on page 46 involves only X_1 and X_2. It is, in fact, true that a common M and σ may be fitted to any two X curves, and the constancy of the ratios $X_2 - X_1 : X_3 - X_1$: etc., is then the condition that this common M and σ shall also fit X_3, etc. The next step is to investigate the condition that must be satisfied if the common M and σ found for two X curves are such that there is a linear relation between M and σ.

Suppose there is in fact the relation $M = a\sigma - b$. Substituting in this the solution given by equations (4), we find

$$-\frac{X_1}{X_2 - X_1} = \frac{a}{X_2 - X_1} - b,$$

i.e. $$-(1 + b)\, X_1 + bX_2 = a.$$

Thus a necessary condition is that there shall be a linear identity between X_2 and X_1. This condition is also sufficient, for if there is a linear identity of the form $lX_1 + mX_2 = n$, then, writing $X_1 = -M/\sigma$ and $X_2 = 1 - M/\sigma$, we obtain the equation

$$-(l + m)\, M = n\sigma - m,$$

and there is a relation of the form $M = a\sigma - b$, the constants being connected by the equations

$$\frac{l + m}{-1} = \frac{n}{a} = \frac{m}{b} \left(= \frac{l}{-b - 1} \right).$$

If we write $n = -a\,(l + m)$ in $lX_1 + mX_2 = n$, we obtain

$$l\,(X_1 + a) + m\,(X_2 + a) = 0,$$

which shows that, when X_2 is plotted against X_1, the straight line that results if the common M and σ for these two items are connected by a linear relation, cuts the line $X_2 = X_1$ at the point $(-a, -a)$. Now a is the constant value of M/σ, when M is measured from absolute zero. This then, apparently, gives a method for testing whether M/σ, measured in absolute units, is constant, and for finding the value of this constant. The method turns out, however, to be far too insensitive for our present purpose. It will be discussed later. There is another fact of more immediate use that emerges from this analysis. The scale used throughout this algebra is, it will be remembered, defined by t_1 as zero, and $t_2 - t_1$ as unit. On this scale absolute zero is at $-b$. This means that if the absolute zero is used, the value of t_1 is b, and that of t_2 is $1 + b$. Hence

$$\frac{t_2}{t_1} = \frac{1 + b}{b} = -\frac{l}{m},$$

and in order to find t_2/t_1 it is only necessary to read the slope of the X_2, X_1 line from the X_1 axis*.

Reverting to the illustrative example given on page 46, it was

* The identity connecting X_1 and X_2 should, of course, be capable of being thrown into a symmetric form. If T_1 be the ratio of t_1 to $t_2 - t_1$, and T_2 the ratio of t_2 to $t_1 - t_2$, then this symmetric form is readily seen to be

$$T_2 X_1 + T_1 X_2 = a.$$

found in that case that the common M and σ that existed satisfied the linear relation $M = \sigma - 60$. It is easily verified that the X's are connected by linear identities. That connecting X_2 and X_1 is $6X_2 - 7X_1 = 1$. This meets the line $X_2 = X_1$ at the point $(-1, -1)$, so that, in accordance with the preceding theory, the constant value of M/σ should be 1, and this is the value that was actually found. Again, t_2/t_1 should be equal to the slope of the X_2, X_1 line, i.e. to 7/6, and this is satisfied by the values found for t_1 and t_2.

This method of scaling the items can be put in another form, somewhat more convenient for our present purpose. If M_n be measured from absolute zero, but in terms of the unit t_n, then, since the constant value of M/σ is clearly unaltered by changing the unit, so long as we are still measuring from absolute zero, we may put

$$M_n = a\sigma_n \quad \text{and} \quad X_n = \frac{1 - M_n}{\sigma_n},$$

and find

$$X_n + a = \frac{a}{M_n}.$$

With regard to the presence now of suffixes to the M's, it should be carefully noted that this does not imply that the hypothesis of common M and σ functions has been forsaken. The M functions are still supposed to be the same, but they are measured now each in terms of its own t, and the numbers representing them will therefore be different. M_1 and M_2 being expressed, then, in terms of t_1 and t_2 respectively, we may, by virtue of the above equations, write

$$\frac{X_1 + a}{X_2 + a} = \frac{M_2}{M_1}$$

in

$$l\,(X_1 + a) + m\,(X_2 + a) = 0,$$

obtaining

$$lM_2 + mM_1 = 0.$$

If, then, these M_2 and M_1 are plotted against one another, the resulting graph should be a straight line through the origin, whose slope measured from the M_2 axis is t_2/t_1.

There emerges the following method of scaling the items in the K test, and of finding the nearest possible common M function, on the hypothesis that the value of $\frac{\sigma}{M}$ obtained by Thurstone's method (i.e. 0·18) is correct. First tabulate the values of M_n, calculated from $M_n = 1/(1 + 0{\cdot}18X_n)$, which follows from $X_n = (1 - M_n)/\sigma_n$ if $\sigma_n = 0{\cdot}18M_n$. By drawing a large number of the M_p, M_q graphs, the values of all the t_n can be found in terms of any one of them, t_1 say.

The values of the M_n function for any item n may then be calculated, in terms of t_1 as unit, from $M_n = t_n/(1 + 0{\cdot}18X_n)$, i.e. by multiplying the values of $1/(1 + 0{\cdot}18X_n)$ already tabulated by the value found for t_n. If there is a common M function, the values obtained for each item will, of course, be identical. As a matter of fact, as was expected, this does not turn out to be the case, but the method gives a very good first shot at the closest possible common M function.

There is another more direct method of looking at this last process. If the values of M calculated from $1/(1 + 0{\cdot}18X_n)$ are really values of the same function, expressed in terms of different units, then the values obtained for any age x will be inversely proportional to the units employed, or, in other words, M_p/M_q should be constant, and equal to t_q/t_p, p and q denoting any two items.

In practice the M_p, M_q curves were not found to be accurately straight lines, and the best straight line through the points was often not through the origin. The method adopted, however, was to draw the best line through the origin. The items were in the first place scaled in sets of similar questions, the first group to be scaled being numbers 60 to 70. For these 21 graphs were drawn, giving the results:

$t_{61}/t_{60}=1{\cdot}03$, $t_{62}/t_{60}=1{\cdot}02$, $t_{61}/t_{62}=1{\cdot}015$, $t_{63}/t_{60}=1{\cdot}09$,
$t_{64}/t_{60}=1{\cdot}24$, $t_{64}/t_{63}=1{\cdot}125$, $t_{65}/t_{60}=1{\cdot}125$, $t_{64}/t_{65}=1{\cdot}09$,
$t_{66}/t_{60}=1{\cdot}21$, $t_{66}/t_{63}=1{\cdot}10$, $t_{64}/t_{66}=1{\cdot}015$, $t_{67}/t_{60}=1{\cdot}24$,
$t_{67}/t_{64}=1{\cdot}01$, $t_{68}/t_{60}=1{\cdot}32$, $t_{69}/t_{60}=1{\cdot}30$, $t_{68}/t_{69}=1{\cdot}045$,
$t_{69}/t_{65}=1{\cdot}13$, $t_{68}/t_{66}=1{\cdot}08$, $t_{70}/t_{60}=1{\cdot}145$, $t_{69}/t_{70}=1{\cdot}13$,
$t_{70}/t_{65}=1{\cdot}005$.

If these are examined, they will be found to be remarkably consistent. The values of t for questions 60 to 70 adopted as best representing these values were

Item	60	61	62	63	64	65	66	67	68	69	70
t	100	103	102	109·5	123	113	121	124	132	128	114

With these values of t, the values of M_n $(= t_n/(1 + 0{\cdot}18X_n))$ were calculated for the eleven items, and the mean values at the several ages found.

Each set of questions was treated in this way, the t's for each group being found in terms of that for the first of the group, which was provisionally called 100. There now arose the question of scaling across from one set of questions to another. This was effected by treating the means of the sets as individual items, and proceeding as before. The t values for the sets were evaluated in terms of that for the set 60 to 70; and they gave, of course, the values of t for the

first items of the groups in terms of t_{60}, which was taken as 100. The individual items could then be scaled in terms of this.

With the provisional scaling thus found, a complete table of the M_n was constructed, and the variations from the means in every question were examined. It was found, for example, that in No. 14 these were $-5{\cdot}2$, $-5{\cdot}9$, $-2{\cdot}3$, $-0{\cdot}9$, $-0{\cdot}1$, $-1{\cdot}4$, $2{\cdot}3$ and $0{\cdot}3$; so that this question had apparently been rated too low, and required scaling up. The whole table was examined carefully in this way, and the scale values of the items adjusted slightly, as required, to make the total algebraic variation for any one item as small as possible. With the new scale values obtained the table given in Appendix II was constructed. In the final adjustment about as much scaling up was needed as the reverse, so that the means were to all intents and purposes unchanged. The adjustments were as a matter of fact small, only exceeding 1 in nine cases, and being in the majority of cases only 0·5. In about 50 per cent. of the items no change was needed. As a result of this adjustment the mean variation was definitely reduced, though only by a small amount.

It might be supposed that by a further fine adjustment the scaling could be carried to a higher degree of accuracy. This is not so, however, for we have already, if anything, slightly exceeded the degree of accuracy that we are entitled to claim, in view of the number of decimal places to which the original values of X were given. The values of $1/(1 + 0{\cdot}18X_n)$ could only be taken to three places of decimals, and therefore the tabulated values of M obtained by multiplying these by a t of the order of 100 could only be taken to one place of decimals, and may be one, or occasionally two, out in this last place. It would seem reasonable to assert, however, that the scaling given is within 0·5 of that which gives the closest common M, on the hypothesis that $\sigma/M = 0{\cdot}18$. While on this point there are two interesting verifications of the scaling that may be worth giving here.

If there is a common M function, and if these values of t give the amounts of the common ability it measures needed to pass the various items, then it is possible to obtain a connection between m, the amount of ability possessed by a child, and the score that he or she should obtain. This score is simply the number of items for which t is less than or equal to m. Table VIII gives the least abilities required to produce the scores shown, the score jumping by 1 when an ability is passed that equals the t for 1 question, and so on. If

these abilities are graphed against the scores, as in Fig. 8, it is found that the points come fairly well on the straight line

$$\text{ability} = \text{score} \times 0{\cdot}74 + 62,$$

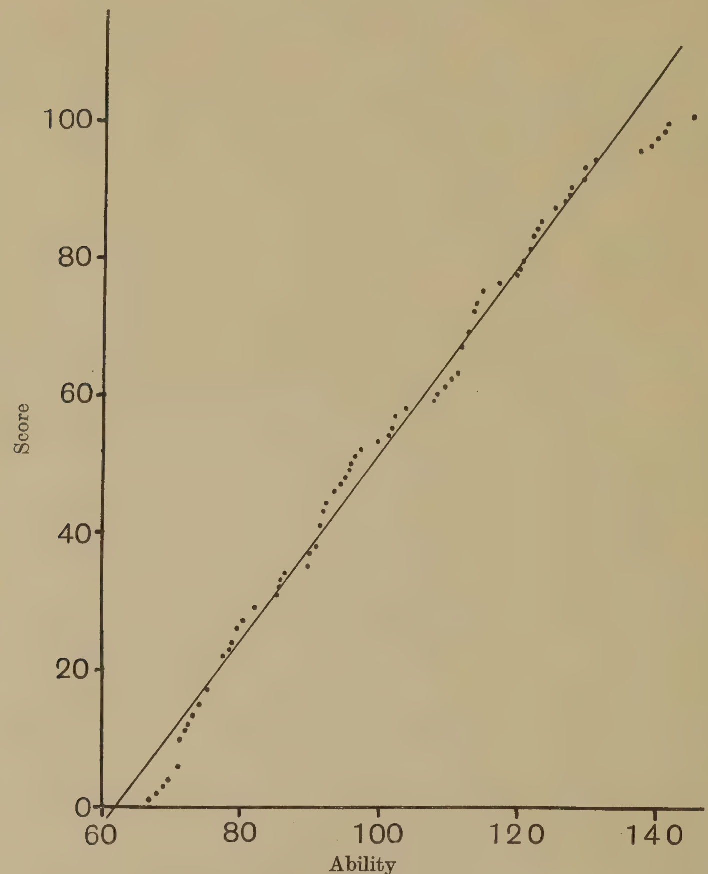


Fig. 8. Plot of *K* test raw score against absolute ability.

except near the beginning and end of the scale. This implies that the questions are distributed fairly evenly over the scale, and that, speaking roughly, unit increase in score means an increase of 0·74 in ability*.

* It was explained at the beginning of Part I that one of the reasons for trying to find an absolute scale was that the value of one unit of raw score might, and generally would, vary considerably from one part of the scale to another. Curiously enough, however, it seems that, in the particular case of the *K* test, the value of one unit of raw score is very nearly the same at all parts of the scale.

Table VIII

Ability	Score	Ability	Score	Ability	Score	Ability	Score
67	1	84·5	30	100	53	121	79
68	2	85	31	101·5	54	122	81
69	3	85·5	32	102	55	122·5	83
69·5	4	86	33	102·5	57	123	84
71	6	86·5	34	104	58	123·5	85
71·5	10	89·5	35	108	59	125·5	87
72	11	90	37	108·5	60	127	88
72·5	12	91	38	109·5	61	127·5	89
73	13	91·5	41	110·5	62	128	90
74	15	92	43	111·5	63	129·5	91
75	17	92·5	44	112	67	130	93
76	19	93·5	46	113	69	131·5	94
77	20	94·5	47	113·5	72	138	95
77·5	22	95	48	114	73	139·5	96
78·5	23	95·5	49	115	75	140·5	97
79	24	96	50	117·5	76	141·5	98
79·5	26	96·5	51	120	77	142	99
80·5	27	97·5	52	120·5	78	145·5	100
82	29						

The interesting point for us at present is that, if the scores corresponding to the mean values of M given at the foot of Appendix II are calculated from the equation given above, the mean scores at the various ages should be obtained. The results of this calculation are:

Age	$7\frac{1}{2}$	$8\frac{1}{2}$	$9\frac{1}{2}$	$10\frac{1}{2}$	$11\frac{1}{2}$	$12\frac{1}{2}$	$13\frac{1}{2}$
Actual norm	23·2	32·0	39·6	45·3	53·2	59·3	63·6
Calculated norm	23·5	31·4	40·1	45·1	53·6	57·3	64·6

Considering the relative roughness of this method of verification, the agreement is remarkable.

Another method that gives an interesting picture of the self-consistency of the results so far obtained is to plot the t_n against the age-assignments of the items given in Appendix II to Part I. These latter, being the ages at which 50 per cent. of the children answer the items should be the ages at which the mean ability is equal to the corresponding t_n. The points plotted in this manner should, then, reproduce the ability curve. The result, with the mean M points marked in, is shown in Fig. 9, and the agreement is again seen to be good.

It may be as well at this point to make it clear that, from the present point of view, there is no exact scaling of the items. The t_n could be adjusted to make the M_n agree exactly at any desired age, instead of choosing them, as has been done, to produce the best general agreement. Attention may also be again directed to the

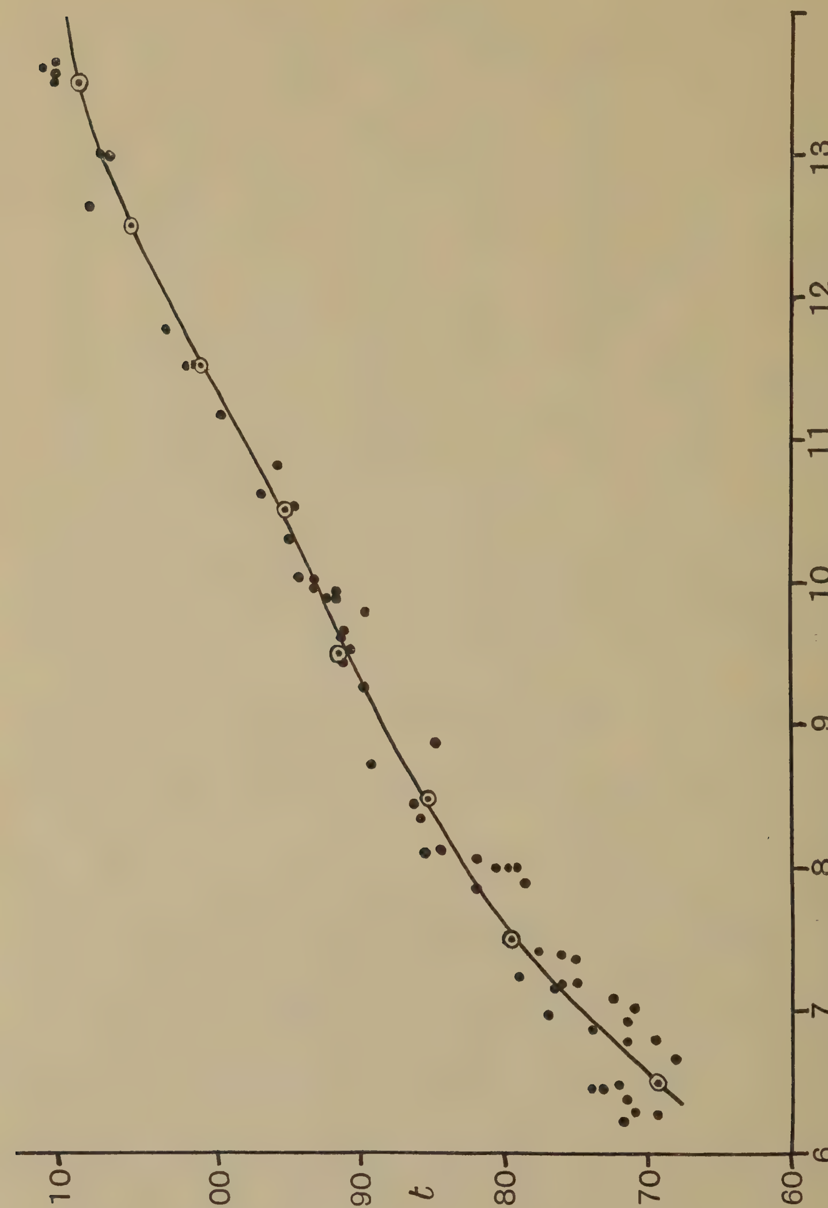


Fig. 9. Comparison of plots of age-assignments of various test items with the mean growth curve.

complete indeterminateness of the problem, previously emphasized. The M_n functions obtained are not put forward as necessarily the correct solution. They need to be carefully examined, and we must be prepared to modify our ideas as we progress, while trying all the time to find the simplest hypothesis that will fit the facts.

The most striking fact that presents itself on examination of the table in Appendix II is the surprisingly small mean variation. The average departure from the mean M function over all the 643 values of M tabulated is only 1·8, i.e. 1·8 per cent. of the ability needed to answer question 60. There is, however, a point of importance in considering these variations that must not be lost sight of. The actual percentage error can be made as small as we please, simply by taking the origin sufficiently far back. Consider the following illustration. If the annual variations in temperature at a place in England were being considered, an error of 15 degrees Centigrade (or 27° Fahrenheit) would be thought a big error, being about 30 per cent. of the annual variation. It would, however, be only about 5 per cent. of the mean temperature on the absolute scale of temperature. So, in the question before us now, a small error must not be lightly dismissed. Rather must we seek to compare it with a variation in ability that for practical purposes would be considered important. For such purposes a variation in ability that is comparable with a year's development in ability cannot be regarded as small. Of course, children are, in practice, quite often placed wrongly by a year, or even more, as regards ability, but this would, presumably, be regarded as a serious misplacing. Now the increases in ability, on the scale of measurement we are using, from 6 to 7, 7 to 8 and so on, are 9·7, 5·8, 6·5, 3·7, 6·3 and 3·4 respectively. It is clear then that even so small a divergence as 1·8 per cent. of t_{60} cannot be too lightly dismissed as negligible. The actual variations for individual items are quite frequently much bigger than this, and are not infrequently comparable with a year's growth in ability. Thus, while the variations are small on the scale of measurement used, they are really of importance, and certainly cannot be passed over.

The question before us at the moment, however, is the justification or otherwise of Thurstone's tacit assumption of a common M function, and we must inquire how far these variations affect his method. The important point in this connection, as explained on page 44, is the variation in the coefficients $\{M(q) - M(p)\}/\sigma(p)$ and $\sigma(q)/\sigma(p)$. The values of these coefficients have been calculated for the case $p = 10$

and $q = 11$, i.e. for that part of the table at which the variations from the mean are least, and are shown in Table IX. To calculate them the second was taken to be $M(q)/M(p)$, and the first to be

$$\{M(q)/M(p) - 1\} \div 0{\cdot}18.$$

Bearing in mind what has already been said about the order to which the calculations may be justifiably taken, it is readily seen that the first coefficient may be one or two out in the second place of decimals, but that the second may be taken to three places.

Table IX

1	0·67	1·120	26	0·36	1·065	51	0·48	1·087	76	0·34	1·061
2	0·53	1·095	27	0·31	1·056	52	0·26	1·047	77	0·35	1·063
3	0·81	1·145	28	0·09	1·016	53	0·31	1·056	78	0·30	1·054
4	0·67	1·120	29	0·31	1·055	54	0·27	1·049	79	0·28	1·050
5	0·44	1·079	30	0·49	1·088	55	0·29	1·053	80	0·42	1·076
6	0·72	1·129	31	0·39	1·071	56	0·32	1·057	81	0·31	1·056
7	0·73	1·132	32	0·29	1·053	57	0·24	1·043	82	0·31	1·056
8	0·22	1·039	33	0·35	1·063	58	0·57	1·102	83	0·23	1·041
9	0·51	1·091	34	0·44	1·079	59	0·17	1·031	84	0·17	1·030
10	0·51	1·092	35	0·48	1·086	60	0·47	1·084	85	0·23	1·042
11	0·51	1·091	36	0·48	1·087	61	0·42	1·076	86	0·02	1·004
12	0·33	1·060	37	0·44	1·079	62	0·41	1·074	87	0·31	1·056
13	0·47	1·084	38	0·28	1·050	63	0·39	1·071	88	0·25	1·045
14	0·28	1·051	39	0·15	1·027	64	0·39	1·071	89	0·26	1·046
15	0·25	1·045	40	0·46	1·082	65	0·37	1·066	90	0·29	1·053
16	0·45	1·081	41	0·25	1·045	66	0·28	1·050	91	—	—
17	0·37	1·066	42	0·83	1·149	67	0·48	1·086	92	0·35	1·063
18	0·19	1·034	43	0·54	1·097	68	0·37	1·067	93	0·37	1·067
19	−0·23	0·958	44	0·48	1·086	69	0·22	1·040	94	0·49	1·089
20	0·31	1·056	45	0·51	1·091	70	0·50	1·090	95	0·27	1·048
21	0·52	1·094	46	0·31	1·056	71	0·32	1·057	96	—	—
22	0·41	1·073	47	0·23	1·041	72	0·29	1·052	97	—	—
23	0·48	1·086	48	0·36	1·065	73	0·31	1·056	98	—	—
24	0·57	1·102	49	0·39	1·071	74	0·33	1·060	99	—	—
25	0·28	1·051	50	0·43	1·077	75	0·32	1·058	100	—	—

The variation in the first coefficient is seen to be considerable, and such as to give rise to grave doubts as to the validity of a result obtained by taking the coefficients to be all equal. On the other hand the variations in the second coefficient are comparatively small, and a result obtained by equating these coefficients may well be a very good first approximation. Now it is the first coefficient that is used in the calculation of M, in Thurstone's method, while the second is used for the evaluation of σ. It might be supposed then that the values of M given by Thurstone's method would be unreliable, while those for σ might be received with more confidence. In this connection it is interesting to compare the mean M function obtained as a result of the scaling process, given at the foot of the table in Appendix II, with the M function that results from the application of Thurstone's

method, given in Part I, Table V, or, with the 5·556 added to obtain the absolute values, in the second column of Table IV of this part. A reference to Part I, Appendix II, shows that the par age for item 60 is 11·16 years. The values of M (Thurstone's method) for $10\frac{1}{2}$ and $11\frac{1}{2}$ years respectively are 6·7292 and 7·1508, and hence it is easy to see that the value for 11·16 years is 7·0103. This, then, is the absolute value of M (Thurstone's method) required to answer item 60, and is thus equal to 100 on the scale used in Appendix II to this part. The ratio of the units employed in the two sets of measurements is therefore 100/7·0103, i.e. 14·23. The following table gives the comparison of the M obtained by Thurstone's method (M') with the M of Appendix II to this part (M'').

Age	$6\frac{1}{2}$	$7\frac{1}{2}$	$8\frac{1}{2}$	$9\frac{1}{2}$	$10\frac{1}{2}$	$11\frac{1}{2}$	$12\frac{1}{2}$	$13\frac{1}{2}$
M''/M'	14·6	14·3	14·2	14·2	14·2	14·2	14·23	14·18

The remarkable agreement demonstrated here, coupled with the relatively small variations of the individual M's from M'', in spite of the comparatively large variations in the coefficient concerned in the calculation of M', suggest an inquiry as to whether the proof of Thurstone's second equation really does depend essentially on the equating of these coefficients. The remarkable fact transpires that Thurstone's second equation can be proved without the assumption of a common M function entering into the argument at all. Throughout the proof, which will now be given, suffixes will always be used to denote the numbers of the items, while the age will be put in brackets in conformity with the usual functional notation. A bar over a letter will denote the average of the quantity concerned taken over the whole set of items in which it occurs. Thus $M_n\,(p)$ will denote the value of the M tested by item n at age p, while $\bar{M}\,(p)$ will denote the average of the $M_n\,(p)$ taken over all the items for which it has been evaluated.

If any function whatever of the age, x, be taken for σ, then the corresponding M_n are given by the equations

$$\sigma\,(x)\,.\,X_n\,(x) = t_n - M_n\,(x).$$

Writing down this equation for two ages p and q, and subtracting, it follows that

$$\sigma\,(p)\,.\,X_n\,(p) - \sigma\,(q)\,.\,X_n\,(q) = M_n\,(q) - M_n\,(p).$$

Summing over all the items concerned, and dividing by the number of items, there results the equation

$$\sigma\,(p)\,.\,\bar{X}\,(p) - \sigma\,(q)\,.\,\bar{X}\,(q) = \bar{M}\,(q) - \bar{M}\,(p),$$

which is identical in form with Thurstone's second equation, and shows that this equation calculates accurately the mean M function

resulting from any given σ function, whether this be obtained from Thurstone's first equation, or otherwise, provided that the σ function is the same for all items.

Thurstone's first equation, that employed in calculating σ, may be proved on similar lines, though not in this case without the assumption of a common M function. As this proof throws considerable light on the method, it will be given here. Assuming a common σ function for all items, it is readily seen that

$$\sigma(p)\,[X_1(p) - X_2(p)] + M_1(p) - M_2(p) = t_1 - t_2,$$

and $$\sigma(q)\,[X_1(q) - X_2(q)] + M_1(q) - M_2(q) = t_1 - t_2,$$

whence

$$\sigma(p)\,[X_1(p) - X_2(p)] + M_1(p) - M_2(p) = \sigma(q)\,[X_1(q) - X_2(q)] + M_1(q) - M_2(q).$$

Keeping the suffixes 1 unchanged in this equation, but writing 1, 2, 3, etc. in turn in place of the suffixes 2, adding the resulting equations, and dividing by the number of items, we obtain

$$\sigma(p)\,[X_1(p) - \bar{X}(p)] + M_1(p) - \bar{M}(p) = \sigma(q)\,[X_1(q) - \bar{X}(q)] + M_1(q) - \bar{M}(q).$$

A similar equation may be written down involving any other item, by changing the suffixes. If, now, there is a common M function, M_1 will be equal to $\bar{M}$ at all ages, and the equation reduces to

$$\sigma(p)\,[X_1(p) - \bar{X}(p)] = \sigma(q)\,[X_1(q) - \bar{X}(q)],$$

with similar equations for all other items. It has, however, already been seen that, in the event of the existence of a common M function, the ratios $X_1 - X_2 : X_1 - X_3 : X_1 - X_4$, etc., are independent of the age. It follows that

$$\frac{\sigma(p)}{\sigma(q)} = \frac{X_1(q) - \bar{X}(q)}{X_1(p) - \bar{X}(p)} = \frac{X_2(q) - \bar{X}(q)}{X_2(p) - \bar{X}(p)} = \text{etc.}$$

Squaring and adding numerators and denominators of these equal ratios, and using the fact that $\Sigma X_n(q) = n\bar{X}(q)$, we find that they are all equal to

$$\sqrt{\frac{\Sigma\,[X_n(q) - \bar{X}(q)]^2}{\Sigma\,[X_n(p) - \bar{X}(p)]^2}} = \sqrt{\frac{\Sigma X_n^2(q) - 2\bar{X}(q).\Sigma X_n(q) + n\bar{X}^2(q)}{\Sigma X_n^2(p) - 2\bar{X}(p).\Sigma X_n(p) + n\bar{X}^2(p)}}$$

$$= \frac{\sqrt{\dfrac{\Sigma X_n^2(q)}{n} - \bar{X}^2(q)}}{\sqrt{\dfrac{\Sigma X_n^2(p)}{n} - \bar{X}^2(p)}}$$

and thus obtain Thurstone's first equation.

It is thus seen exactly how the assumption of a common M function enters into the proof of this equation. The effect of this assumption may be divided into two parts: (a) the neglecting of $M_1 - \bar{M}$, etc., and (b) the treatment of the ratios $\dfrac{X_1(q) - \bar{X}(q)}{X_1(p) - \bar{X}(p)}$, etc. as all equal. Let us consider these two separately.

With regard to (a), it is, of course, really the differences

$$[M_1(p) - \bar{M}(p)] - [M_1(q) - \bar{M}(q)], \text{ etc.,}$$

that are neglected. Some idea of the error involved may be obtained by evaluating this expression for the various items on the basis of the values given in the table of Appendix II to this part, and comparing its values with those of the other terms in the equation. In Table X, column 1 gives the number of the item, n; column 2 gives the value of $\sigma(10)[X_n(10) - \bar{X}(10)]$; column 3 gives $\sigma(11)[X_n(11) - \bar{X}(11)]$; and the last column gives

$$[M_n(10) - \bar{M}(10)] - [M_n(11) - \bar{M}(11)].$$

For the purposes of the calculation σ has, of course, been taken to be $0{\cdot}18\bar{M}$. It has not been thought necessary to give the results for every item, but the eight questions included in the table are typical. In the majority of items the neglected term (4th column) is small compared with the other two terms, except for a few items in the middle of the test, where the X's are small and the neglected term becomes of the same order as those retained.

Table X

1	−26·18	−28·92	−2·9
3	−21·72	−27·26	−6·8
24	−24·12	−26·90	−3·2
42	− 9·36	−12·81	−3·4
51	− 4·21	− 6·41	−1·9
52	− 1·03	1·83	1·7
62	5·41	4·21	−0·7
71	14·17	12·8	0·8

With regard to (b), it is merely a matter of evaluating the ratios concerned for each item, and testing how nearly they are equal. In Table XI the first column gives the number of the item, the second $X_n(10) - \bar{X}(10)$, the third $X_n(11) - \bar{X}(11)$, while the last gives the ratio $\dfrac{X_n(11) - \bar{X}(11)}{X_n(10) - \bar{X}(10)}$.

The value of $\sqrt{\dfrac{\dfrac{\Sigma X_n^2\,(11)}{n} - \bar{X}^2\,(11)}{\dfrac{\Sigma X_n^2\,(10)}{n} - \bar{X}^2\,(10)}}$ is 0·97.

Too much notice must not be taken of the erratic values obtained in the middle of the table, for here both the numerator and denominator of the ratio concerned are small, and a very uniform result in the

Table XI

1	−1·525	−1·58	1·04	33	−0·615	−0·59	0·96	65	0·905	0·82	0·91
2	−1·210	−1·27	1·05	34	−0·260	−0·33	1·27	66	1·215	1·20	0·99
3	−1·265	−1·49	1·18	35	−0·365	−0·45	1·23	67	1·625	1·38	0·86
4	−1·285	−1·43	1·11	36	−0·615	−0·69	1·12	68	1·875	1·74	0·93
5	−1·285	−1·28	1·00	37	−0·365	−0·42	1·15	69	1·655	1·70	1·03
6	−1·295	−1·47	1·14	38	0·025	0·04	1·60	70	1·065	0·82	0·77
7	−1·265	−1·45	1·15	39	0·335	0·50	1·39	71	0·825	0·81	0·98
8	−1·155	−1·00	0·87	40	0·360	0·23	0·64	72	0·775	0·79	1·02
9	−1·160	−1·20	1·03	41	−0·235	−0·14	0·60	73	0·770	0·76	0·99
10	−1·225	−1·27	1·04	42	−0·445	−0·70	1·57	74	0·950	0·83	0·87
11	−1·205	−1·25	1·04	43	−0·295	−0·44	1·49	75	0·965	0·92	0·95
12	−0·855	−0·80	0·84	44	−0·215	−0·32	1·49	76	0·855	0·80	0·94
13	−1·115	−1·14	1·02	45	−0·320	−0·44	1·37	77	1·175	1·09	0·93
14	−1·105	−1·00	0·90	46	−0·375	−0·33	0·88	78	1·415	1·38	0·97
15	−0·935	−0·81	0·87	47	−0·225	−0·12	0·53	79	1·295	1·29	1·00
16	−1·030	−1·05	1·02	48	−0·095	−0·11	1·16	80	0·805	0·73	0·91
17	−1·055	−1·02	0·97	49	−0·090	−0·13	1·44	81	1·005	0·98	0·97
18	−1·275	−1·10	0·86	50	−0·315	−0·37	1·17	82	1·355	1·32	0·98
19	−1·750	−1·25	0·71	51	−0·245	−0·35	1·43	83	1·695	1·74	1·03
20	−1·415	−1·31	0·93	52	−0·060	0·01	−0·17	84	1·745	1·86	1·07
21	−1·575	−1·60	1·03	53	−0·275	−0·23	0·84	85	1·275	1·31	1·03
22	−1·075	−0·98	0·91	54	−0·145	−0·08	0·55	86	1·605	1·90	1·18
23	−1·505	−1·69	1·12	55	0·660	0·665	1·01	87	0·805	0·79	0·98
24	−1·405	−1·47	1·05	56	0·225	0·23	1·02	88	0·625	0·68	1·09
25	−1·465	−1·34	0·91	57	0·875	0·93	1·06	89	1·645	1·64	1·00
26	−1·125	−1·08	0·96	58	0·805	0·53	0·66	90	1·415	1·38	0·98
27	−0·535	−0·49	0·92	59	0·745	0·88	1·18	91	—	—	—
28	−0·420	−0·17	0·40	60	0·200	0·07	0·35	92	1·400	1·30	0·93
29	−1·715	−1·60	0·93	61	0·505	0·34	0·67	93	1·730	1·59	0·92
30	−1·495	−1·50	1·00	62	0·315	0·23	0·73	94	1·830	1·55	0·85
31	−1·185	−1·15	0·97	63	0·715	0·61	0·85	95	1·755	1·75	1·00
32	−0·640	−0·57	0·89	64	1·590	1·43	0·90	96	—	—	—

value of the quotient is not to be expected. A first glance through the table would possibly give the impression that the variations in the value of the ratio are rather large, especially when it is remembered that the value 1·1 would mean σ decreasing by about 10 per cent., while the value 0·9 would mean it increasing by about 10 per cent., from age 10 to age 11, and these values (0·9 and 1·1) are by no means extreme values among those occurring in this table. A single item is, however, very unreliable, and the important thing to consider is the distribution of the individual values about the mean. On plotting them it is found that there is a very marked central

tendency, with a median value of 0·98, and a quartile deviation of only 0·08. There is then, allowing for variations that may be expected from item to item, good evidence to support the values of σ given by Thurstone's first equation, as being likely to determine a σ function that will give the closest possible common M function, and that this common M function will not differ greatly from the functions for the individual items, as indeed has been found to be the case.

This is confirmed by a geometrical interpretation that may be given to the method, which throws an interesting sidelight on the process, and which also gives a much more rapid, if somewhat cruder, method of calculating M and σ. The equation,

$$\sigma(p) . X_n(p) - \sigma(q) . X_n(q) = M_n(q) - M_n(p),$$

already used on page 57, shows that if there are common M and σ functions, then the $X(p)$ and $X(q)$ plotted against one another for all items will give a straight line, and that if this line is

$$X(p) = m . X(q) + c,$$

then $\sigma(q)/\sigma(p) = m$ and $M(q) - M(p) = c . \sigma(p)$.

These equations may clearly be used exactly as in Thurstone's method to calculate M and σ, in fact a little consideration shows that the two methods amount to very much the same thing, essentially. The ratios $\dfrac{X_1(q) - \bar{X}(q)}{X_1(p) - \bar{X}(p)}$, etc., given on page 60, are the slopes of the lines obtained by joining the individual $X(p)$, $X(q)$ points to the centre of gravity of them all, and Thurstone's method gives the correct way of averaging these slopes to obtain the slope of the line that is statistically the proper one to take. The method given here, of drawing by eye the best straight line, amounts to a weighting of the items that is to a slight extent dependent on the observer, and is not based on the probable error of the X's but rather on the actual variation from all causes that is exhibited from item to item. The results obtained by determining M and σ by this method are:

Age	6	7	8	9	10	11	12	13
M	−0·0238	0	0·472	0·996	1·311	1·768	2·129	2·474
σ	0·952	1	1·08	1·204	1·354	1·381	1·436	1·458

The M values are all somewhat larger than those given in Table V, Part I, but on plotting the two sets there is almost exact linearity exhibited, apart from the 6-year-old point, and little reliance can be placed on this, as the straight line from which it is determined is not

so well defined as in the other cases. When the M and σ are plotted against one another there is a reasonable approach to linearity, giving the result $\sigma/M = 0{\cdot}20$, which is slightly larger than that yielded by Thurstone's method, as explained in Part I. The important point really, which confirms the view already formed of the accuracy of the approximation contained in Thurstone's first equation, is the very close approach to linearity (except in the case $p = 6$) of the X (p), X (q) plots.

As a final confirmation of this value of σ/M, we may say a few words about the X_1, X_2 curves already discussed on page 48, and round off the theory in this respect. The theorem proved there is that, if the X's for two items are plotted against one another, the resulting graph is a straight line if, and only if, the common M function that can be constructed for them is such that there is a linear relation between M and σ. It was also proved that the line meets the line bisecting the angle between the axes in the third quadrant at the point $(-\,a, -\,a)$, where a is the constant value of M/σ when M is measured from the absolute origin defined by $\sigma = 0$. A slightly curved graph might be interpreted as indicating a changing value of M/σ, defined by the intersection of tangents to the curve with the line $X_1 = X_2$. The difficulty with regard to practical application is that, while the curves are nearly straight lines in most cases, they are only inclined at small angles to the line $X_1 = X_2$. The point of intersection is, under these circumstances, difficult to fix, even roughly, and a slight curvature produces big changes in the value of M/σ. A typical graph, that for X_{60}, X_{61}, is shown in Fig. 10. The line drawn is that through the point $(-\,5{\cdot}556, -\,5{\cdot}556)$—which would give the value $0{\cdot}18$ for σ/M—whose slope is t_{60}/t_{61}. The difficulties referred to above are at once apparent here, and it is quite impossible to obtain any independent fact from these curves. The line drawn does, however, seem to fit as well as any other, and in this respect the curves are of service, as confirming the value of σ/M already found. A large number of them have been drawn, and very few of them fail in this respect.

This completes our analysis of Thurstone's method. It will be remembered that this was undertaken originally because the method is only an approximation, and in any such case it is important to investigate the result obtained, with a view to discovering the extent of the approximation and consequently finding out what degree of reliance can be placed on the method. After a very full analysis—

but none too elaborate, considering the importance of the question involved—there can be no other conclusion but the complete vindication of Thurstone's method as giving what is very nearly the closest possible approach to a common M function, and, coupled with the method of scaling here developed, the individual M_n functions that produce this nearest approach to uniformity.

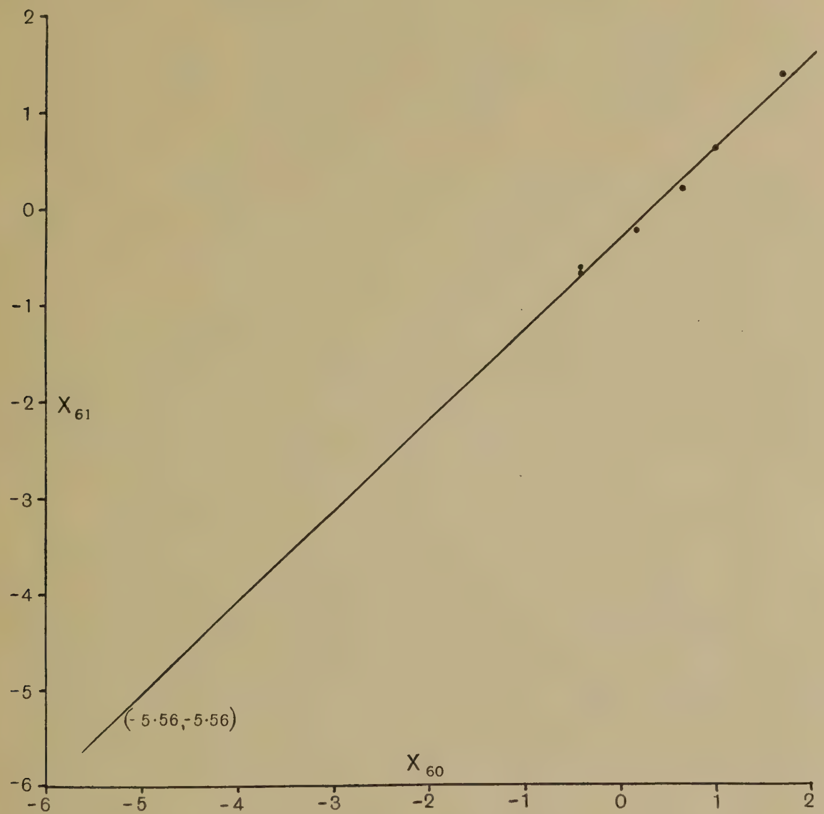


Fig. 10. Plot of X_{60} against X_{61}.

The problem that confronts us now is the question as to whether the solution obtained is really the correct description of the growth in ability in the average individual—bearing in mind the fact, which cannot be too often stressed, that the complete indeterminateness of the equations available makes it possible to construct solutions in infinite variety. A definite decision on this point is, of course, quite impossible in the absence of data showing the performance of the

same set of children for the same test, set at different ages. Without such urgently needed information, the decision arrived at on the question under review can but reflect one's own personal preferences. There are, however, some points that would seem of importance in this connection, and they will be discussed here.

Mention has already been made of a certain uniformity that seems to pervade the irregularities that exist from item to item. This point is well illustrated in the X curves given in Figs. 5 and 6. In the curves for items 42 to 49 there occurs at age 10 what may be conveniently described as a 'kink.' This may, of course, be due to a certain inferiority in the age-group concerned, or to some other external cause affecting this group. If this were the case, however, one would expect to find the same phenomenon occurring in all items. This is not the case. In items 71 to 79 there is no such kink at age 10, but there is on the other hand a similar phenomenon occurring at age 12. The existence of such irregularities always at the same age for the same type of question, but at ages varying from 10 to 13 for different types of questions, suggests the possibility that they are due to peculiarities in the development of the abilities tested by the various items. Facts which support this view and suggest an explanation of the phenomenon will be considered later. For the moment let us accept it as a fact, and consider the position.

The one thing that is certain is that the M_n curves for the various items are all distinct. We have found the set which probably comes nearest to uniformity, and have seen that the divergences from the mean in this case cannot be neglected. It has also been seen that, in addition to random variations, probably due to various accidental and external causes, there exist certain uniformities in the irregularities in the X curves, which will, of course, be reflected in a more or less sharp degree in the M_n curves, however these are constructed from the X curves. Under these circumstances is it probable that the solution found, which gives the most uniformity in the M_n functions, is the correct one? The answer to this question is possibly, one might almost say probably, in the negative. For, if there are causes which produce irregularities in the M_n functions for different items, then it is hardly to be expected that they will adjust themselves so as to produce the greatest possible uniformity in these functions. On the other hand it is unlikely that the ability curves for questions of a similar nature will diverge very considerably from one another, though a certain caution is needed in this respect, for the same

question, worded in the same way, will produce different results in different settings. On the whole the most reasonable conclusion would seem to be that, while the actual state of affairs is perhaps not exactly that given by the M_n functions we have calculated, it probably does not differ very markedly from these. The general tendencies exhibited by the solution found are probably not far from correct, but it might reasonably be expected that the 'kinks' discussed in the preceding paragraph are somewhat more accentuated than in the solution actually found. In this connection it is of interest to consider briefly another method, by no means less likely than the one adopted, of calculating the individual M_n from the results obtained by Thurstone's method.

It has been proved on page 57 that, the value of σ having been calculated from Thurstone's first equation, the second equation gives exactly the mean of the resulting M_n, if these are calculated from $M_n = t_n - \sigma X_n$. It is the resulting mean M to which σ stands in the ratio 0·18. It would, perhaps, seem more natural to calculate the M_n from this equation, rather than to use the method that has actually been adopted. In order to do this each M_n would have to be measured, in the first instance, from its own t_n as zero—the unit measure of ability being of course that adopted in the calculation of σ, i.e. σ (7) in our case. The M_n would then be given by $M_n = -\sigma X_n$. These calculated, it would be possible to scale the items on the basis of finding the nearest common M function, and to find the t_n on the scale $t_1 = 0$, simply by taking t_n equal to the mean value at all ages, of $M_n - M_1$. It is quite unnecessary to carry out this scaling process however. A change in the value of t_n would only have the effect of a change in origin, just as a change in the value of t_n in our original method of scaling is only equivalent to a change of unit—in neither case is the form of M_n affected. This second method of calculating the M_n would not give a result so near uniformity as the original method, because the nature of the formula from which the M_n are calculated reflects the irregularities in the X's more markedly than if the M_n are obtained from $1/(1 + 0{\cdot}18X_n)$, over the greater part of the range of values of X that occur. For this very reason the kinks already referred to will be somewhat more marked in this second method of calculation than in the first. There is, however, no cogent reason for adopting the original method of calculating the M_n in preference to that now suggested. Indeed it might possibly be argued that it is more reasonable to assume that the spread is a definite

function of the age, rather than to assume that it is in a constant ratio to the ability, and therefore varying to a certain extent from item to item, at any given age, with the M_n. It is, however, in the absence of further information, entirely a matter of personal preference, and it is not intended here to give an opinion one way or the other. The difference between the M_n functions obtained is shown

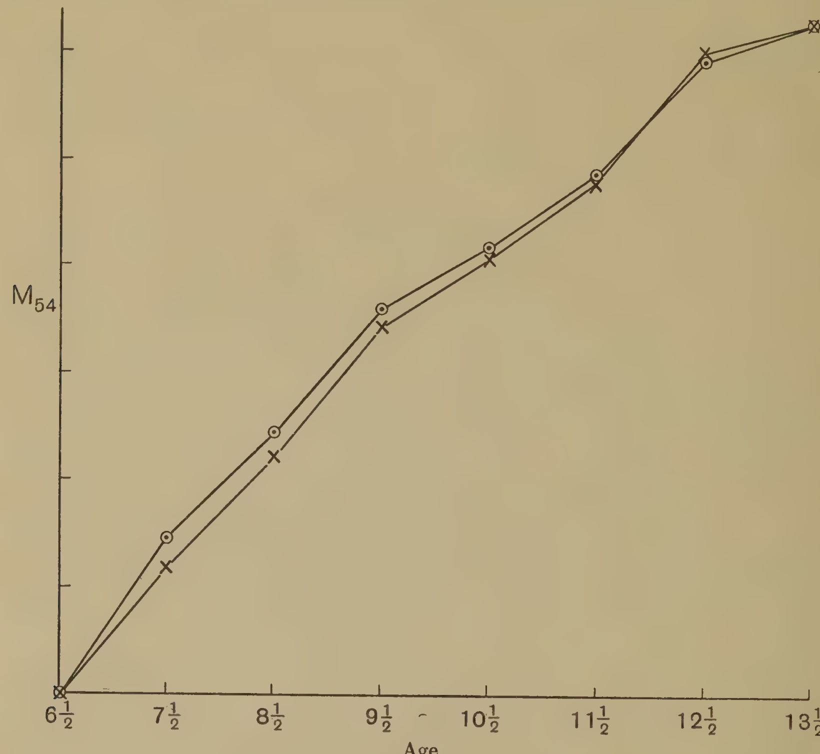


Fig. 11. Comparison of growth curves obtained from item 54 by alternative methods.
⊙ denotes M_{54} obtained from $-\sigma X_{54}$.
× " " " " $\dfrac{1}{1+\cdot 18\, X_{54}}$.

clearly in Fig. 11, where M_{54}, as calculated by the two methods, is shown. This item has been chosen as example because, while the dip at ages 11 and 12 is well marked, it is really quite a mild case, and not such as would give an exaggerated idea of the phenomenon.

Which of these two methods of calculating the M_n functions from the results yielded by Thurstone's method is adopted is really immaterial. There is not very much difference in the final results

obtained, and the point is not worth further discussion, since, as has already been stated, we are not in a position to say more than that the true M_n curves are probably fairly closely of the type obtained, but with the variations of a regular nature possibly more marked.

It is now necessary to examine more closely the agreement with the G curve that has been seen to hold to a first order. If Fig. 3, giving 77 of the items plotted against the standard curve, be examined, and a mean curve drawn through the points, this will be found to dip decidedly below the standard curve between the isochronic ages 1·3 and 1·7, and to rise distinctly above it shortly after. These facts are of course the expression, on that graph, of the divergences from the G curve of a regular nature that have already been noted. If in the true M_n curves these divergences are much more marked, it would not be possible to say that the G curve is followed very accurately. The question is, can the G curve hypothesis be modified to fit such a state of affairs, or would it have to be abandoned? It will be seen in the sequel that not only is this modification possible, but that the solution of the problem thereby obtained, while being a little more involved, and somewhat less exact, appears to give a much more rational view of the growth of intelligence.

It was remarked on page 24, when the G curve was first introduced, that it is followed 'given reasonably constant conditions of nurture.' Under any change in these conditions, a modification of the law of growth is of course to be anticipated. Among such changes are to be included such effects as 'coaching,' and physiological developments, healthy or morbid. As regards the first, the M curves for some of the earlier questions in the K test, such as the simple arithmetical and language questions will very possibly be modified in this way at the earlier ages. Children round about 6, 7 and 8 do in practice receive a certain amount of drill in such matters, and it is not unlikely that the average ability in respect of them may, under the influence of this, show a more rapid rise than is the case in some of the other types of questions. Exact analysis of such effects is impossible, and we pass on to the examination of the second possible cause of deviation.

There has been in recent years a steadily growing mass of evidence as to the importance of the endocrine glands in the growth of the individual, and the big effect they have in determining his or her mental and physical characteristics*. The hyperthyroid case is, for

* See, for example, L. Berman, *The Glands Regulating Personality*; E. A. Schäfer, *The Endocrine Glands* (1916); H. Crichton-Miller, *B.M.J.* Sept. 1922, p. 551.

instance, frequently found to be a person of aesthetic temperament, with a development of creative power in literature, the arts or kindred spheres, beyond the average. The more active suprarenal glands, that produce, apparently, the keen business type, produce also a tendency to the 'business man's disease'—high blood pressure. These are but two illustrations from the vast amount of evidence that exists to show the importance of these glands in determining mental, as well as physical, growth and characteristics. Now it is well known that the functioning of these glands undergoes modification before and during puberty. Some, the thyroid for example, become more active, while others regress. It is, in fact, not unlikely that they control, to a considerable extent, the physical developments that take place in the individual at this period. In view of this fact, and of the close relation between their state of development and mental characteristics, it would seem not unreasonable to suppose that there exists at this period what amounts, in effect, to a change in nurture of the brain, sufficient to account for the suggested divergence from the growth curve. Considering also the relation of certain glands to certain types of mental activity, it is perhaps to be expected that the developments occurring at this period will affect different types of ability in different ways, and to different extents. Before passing on to an attempt at analysing the M_n curves from this point of view, it may be remarked that the evidence on the physical side for some such effect round about these ages as that outlined is so strong, that it almost constitutes good presumptive evidence for regarding the variations under discussion as reflecting a real phenomenon of growth, and not as being in any way spurious.

The effect produced by the influences described in the preceding paragraph might be described mathematically in a variety of ways. The change in 'nurture' might produce simply a transition from one G curve to another, or on the other hand the resulting growth curve may be obtained by the superposition on the general growth curve of another curve which becomes effective only at the ages concerned. In considering the possibility of such superposition it will be supposed that the two curves combined are G curves. This law has been retained for two reasons. In the first place it would seem to be, in the light of the analysis on page 34, an eminently reasonable law of growth, and again, it has been found by Dr Courtis to describe accurately growth in many fields. There would seem to be no *a priori* reason for combining the two curves by addition rather than by multiplication, but

it has, as a matter of fact, been found impossible to produce the observed effects by the multiplication of two G curves, whereas the attempt at decomposition of the actual growth curve into the sum of two G curves met with striking success. Before giving actual figures, it is interesting to consider in a general way the limitations on the constants of the two component curves, if their sum is to produce the desired effect.

If a quantity y follows a G curve, then dy/dx ($x =$ time) will evidently follow a bell-shaped curve somewhat skewed towards the left. If we combine by addition two such curves for dy/dx (see Fig. 12 *a*) we get a double-humped curve of the shape shown in Fig. 12 *b*, provided that (1) the peaks of the two component curves

Fig. 12.

are not too close together; and (2) the right-hand curve rises to its peak more gradually than the left-hand curve initially (i.e. until well past the maximum of the left-hand curve). If the two maximum points are too close together, or if the right-hand curve rises relatively steeply initially, and has a broader summit, the composite curve will have but one peak.

Now the position of the peak of each component curve depends on the value of T, the time at which the y curve reaches its point of inflexion; and a more gradual initial rise of the dy/dx curve and a sharper peak are associated with a larger T and a smaller θ.

Next let us consider the effect of increasing the scale of the left-hand curve relative to that of the right-hand curve. This is equivalent to increasing the value of C (the maximum value of y) for the left-hand curve, relative to that for the right-hand curve. Combining the two curves we shall then get a result of the kind shown in Fig. 13 *a*, *b*, provided the C's are in a suitable ratio.

Now Fig. 13 *b* corresponds exactly to the state of affairs on our M curves, the observed portions of which are beyond the main points of inflexion, namely a steadily decreasing slope, then a small temporary increase of slope, and finally a steady diminution. These changes of slope produce the kink.

Hence it follows that the combination by addition of two G curves, with constants C_1, T_1, θ_1, and C_2, T_2, θ_2, will produce the observed results provided that T_2 and θ_1 are respectively greater than T_1 and θ_2, and secondly that C_1 and C_2 are in a suitable ratio, C_1 being greater than C_2. This is, however, exactly the type of combination suggested by the physiological aspect of the problem.

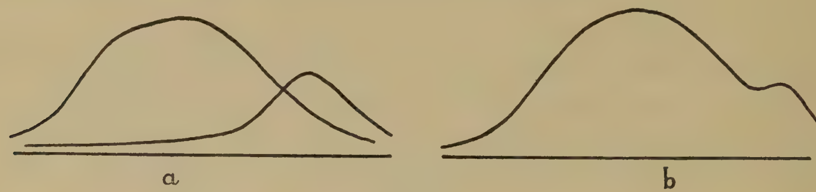


Fig. 13.

We have then to see whether the observed M curves can be reproduced by the combination of a G curve with a relatively low T and a comparatively large θ, with a G curve whose T is about 11 years, whose θ is small and whose maximum is less than that of the first. Rough numerical trials suggest that the actual curves can be reproduced very closely in this way, and an attempt was made to see how accurately the decomposition could be made in a particular case, with very marked success. The item chosen for this work was No. 54, and it was found that the calculated values of the M function could be reproduced very closely indeed by the addition of (1) a G curve with $T = 6\frac{5}{6}$, $\theta = 1\frac{2}{3}$, with (2) a G curve with $T = 11\frac{1}{6}$, $\theta = \frac{5}{12}$, the maximum in the second being one-quarter of that in the first. The equation of such a curve would be

$$y = e^{-e^{-\frac{x - 6\frac{5}{6}}{1\frac{2}{3}}}} + \tfrac{1}{4}e^{-e^{-\frac{x - 11\frac{1}{6}}{\frac{5}{12}}}} = u + \tfrac{1}{4}v, \text{ say.}$$

The values of this function are given in the last column of Table XII. In the first column of that table is the age, in the second M_{54} calculated from $-\sigma X_{54}$, and in the third M_{54} obtained from $1/(1 + 0{\cdot}18X_{54})$. In each case the scale has been adjusted to obtain exact agreement at the two extreme ages. The agreement is seen to be remarkably close.

Another point of considerable interest is that, if these same two G curves be combined with maxima in different ratios, and the resultant curves drawn, forming, of course, the family whose equation is $y = u + kv$, the M curves for the whole set of questions 50 to 59 are reproduced closely in type. Corresponding to curves with a larger

Table XII

6+	0	0	0
7+	0·725	0·6035	0·647
8+	1·220	1·114	1·268
9+	1·800	1·717	1·734
10+	2·090	2·033	2·038
11+	2·425	2·393	2·392
12+	2·955	3·010	2·992
13+	3·140	3·140	3·140

dip are those with a bigger value of k. The same is true of the other sets of similar items, but the component curves vary to some extent from one set of questions to another. Speaking generally, it seems that the second of the two curves has a somewhat higher T in the case of questions of a harder type than those earlier in the paper. Also the kink would appear, on the whole, to be more marked in the easier questions than in the harder ones of the same set. Too much stress should not be laid on the exact decomposition of M_{54} given above, but rather on these remarks of a more qualitative nature. As a matter of fact considerable latitude is possible in the choice of the first G curve, which, combined with a second one of near the type discussed, will produce fairly accurately the observed M curves. Also it is hardly to be expected that the effect superposed at puberty is of the simple type described by a single G curve. Closer agreement could, no doubt, be produced if an attempt were made to build up the actual M curves by a combination of more than two G curves, but it would be manifestly absurd to attempt to extract this degree of accuracy from the data.

It is very probably impossible to reduce the growth of ability to strict mathematical law, but a more general picture of the process should no doubt be formed. The effect of the endocrine glands is not to be regarded as confined to the age of puberty. They are active in the determination of personality throughout life. At some ages certain of them are more important than others; at other ages the position may be reversed. Thus the thymus and the pineal regress during childhood. These changes are not constant in character from individual to individual. Some interesting examples of the effect on the personality of the subject of abnormalities in the functioning of these glands are contained in the references given on page 67. Moreover, it would be idle, in the present position of the physiology

of growth, to pretend to take account of all the factors present at any given stage in life. All things considered, it would seem impossible to regard the development in intelligence as represented by any simple regular curve. We should rather expect that, as the child grows, so varying influences are called into play and the 'intelligence contour' of the individual unfolds, not continuously, but in a series of spurts, as it were, first in one direction and then in another. Such irregularities in physical growth we are familiar with, and speak of 'springing-up' periods, and 'filling-out' periods. (See Dr H. A. Harris, reported in the *B.M.J.*, April, 1931, p. 583.)

Under the circumstances, to suggest a single curve as representing the development of ability is risky, especially if it is obtained by extrapolation from the period of puberty, when it is only to be expected that perturbing influences are present. There is, however, an important point, already referred to on page 55, that must be taken into consideration in this connection. If the mathematical solution of the problem that has been obtained here is anywhere near the correct one, then differences of ability that are commonly considered large are really nothing like so great as would be expected. The average growth of ability, from 12 to 14, say, has been seen to be small on our absolute scale. Again, one might reasonably expect to find quite a big effect at puberty, yet there is only a relatively small modification of the curve. These facts being taken into consideration we may perhaps still, especially in view of the eminent reasonableness of the figures, regard the curve given on page 31 as representing fairly the average development in all-round ability.

CONCLUSION

In conclusion, we may briefly consider what light is thrown by our results on (1) the nature and validity of Thurstone's method; (2) the nature of 'intelligence'; and (3) the use of intelligence tests as instruments of scientific measurement.

As regards the first of these, it is clear that, in so far as Thurstone's method involves the assumption of M and σ functions which are common to all test items, it is not entirely justified, but it does in fact provide a solution defining the M and σ functions which will be most uniform from item to item, and the deviations from uniformity are small. The M calculated by the method is, indeed, a mathematically accurate average of the M's for the various items if a common σ

function be assumed; and it is an empirical fact that the σ obtained by the method is at any rate close to the average of the σ's, though the proof of Thurstone's equation for σ depends on the assumption of a common M function as well as a common σ function. We might sum up by saying that the M and σ values obtained by Thurstone's method provide a picture of the growth and variability of *all-round* mental ability which is probably not far from the truth, though it must be remembered that the solution of the problem is really indeterminate.

Passing to the second of our points, it is certain that the growth curves of the abilities tested by different test items are distinct from one another, though, if the curves we have obtained are somewhere near the true ones, they do in fact cluster fairly closely about a mean growth curve. Consideration of the latter confirms the hypothesis that σ/M is a constant equal to about 0·18. There is evidence that σ/M is not only nearly constant from age to age, but also that it does not vary very much from item to item. If the value 0·18 requires amending it is probably slightly in an upward direction. The mean growth curve obtained is an almost perfect G curve. Extrapolation on this curve indicates a $6\frac{1}{2}$ per cent. development in all-round ability at birth. The point of inflexion in the curve is at about 4 years. Development reaches 96 per cent. by 18 years of age and 99 per cent. at 24. It must be remembered, however, that this extrapolation is distinctly speculative in character, especially in the downward direction. The growth curves for the separate items are also nearly G curves in form, but most of them exhibit a slight but regular kink somewhere between $9\frac{1}{2}$ and $13\frac{1}{2}$ years. It is possible to explain the deviations of the curves from one another and also the uniform irregularities in particular curves on the hypothesis that ability in any given test item is the result of a combination, by addition, of factors each of which follows, in its growth, a perfect G curve, the constants of these curves varying suitably from one factor to another.

It follows from the fact that the mean growth curve is a G curve, that the curve of all-round growth for any particular individual will also be a G curve, though the constant C for the curve will of course vary considerably from one individual to another.

Finally, what are we to say about the use of intelligence tests as instruments of measurement? It should be clear that, if any such test is to be regarded, not merely as a rough means of placing children in order of ability, but as a scientific scale which can lay claim to a

reasonable degree of accuracy, it must be subjected to some such analysis as has been described in the foregoing pages. One striking point which has emerged in the course of that analysis is that from 6 years onward we are dealing with the upper half of the development curve which has reached by 11 years a level of over 80 per cent. of maturity. Hence the differences we are accustomed to measure with our tests, between one year and another, and one individual and another, though important from a practical point of view, will often be quite small when viewed against the background of the absolute scale. In other words a difference which appears very considerable between two children of 11 (say) may perhaps be quite small when considered as a percentage of the abilities of the children measured from absolute zero. From this standpoint it appears, then, that intelligence tests are actually rather delicate instruments capable of measuring differences which are really slight, and the fact that, in using them, we are often dealing with differences which, on the absolute scale, are comparatively small, may explain the perplexing irregularities and discrepancies which we so frequently meet in our investigations. Viewed in the proper perspective, indeed, one of the most encouraging results of the present inquiry is the fact that, in spite of the innumerable disturbing causes of all kinds which must inevitably interfere with such an experiment as that described in Part I, the results obtained, not only from the scale as a whole, but also from separate items, both individually and in comparison, exhibit such a remarkably high degree of uniformity and self consistency.

APPENDICES

APPENDIX I

PERCENTAGE LEVELS OF DEVELOPMENT AT DIFFERENT AGES GIVEN BY THE VARIOUS ITEMS

Item	5	6	7	9	10	11	12	13	14	15	16	17	18	20	21	22
Age																
6+	54	57	59	58	54	55	56	52	52	55	53	50	57	51	57	65
7+	63	65	67	65	61	63	66	60	60	64	63	60	65	59	67	76
8+	68	68	70	71	67	71	72	67	67	71	71	66	68	62	73	81
9+	74	71	73	75	74	77	76	74	73	77	77	73	71	69	81	84
10+	77	72	75	78	77	79	79	77	77	80	80	76	73	71	84	87
11+	83	81	84	85	83	86	83	83	82	83	86	81	75	75	92	92
12+	85	87	89	90	88	89	85	89	88	88	91	89	79	78	93	92
13+	89	89	91	90	91	92	91	88	89	91	91	87	83	82	93	93

Item	23	24	26	27	29	30	31	32	33	34	35	36	38	39	40	41
Age																
6+	58	59	60	67	56	57	57	55	52	56	54	57	61	68	55	60
7+	67	69	69	75	68	67	65	64	65	62	61	67	66	74	61	67
8+	74	75	76	77	77	75	73	72	72	67	66	73	71	79	66	71
9+	81	82	80	81	84	81	78	76	76	72	72	79	77	82	71	76
10+	83	81	81	83	86	84	80	79	78	76	76	78	79	84	75	80
11+	90	88	87	87	91	91	86	83	83	82	83	85	83	86	81	84
12+	93	91	88	91	93	93	87	87	87	85	88	89	86	88	86	87
13+	94	93	91	93	92	92	90	90	91	90	89	91	90	91	90	91

Item	42	43	44	46	47	48	50	51	52	53	54	55	56	57	58	60
Age																
6+	54	56	54	55	56	56	52	52	56	53	55	58	59	62	59	52
7+	60	63	60	62	60	62	58	59	63	60	61	61	66	67	67	58
8+	65	67	63	67	68	68	65	64	69	66	67	69	73	71	73	62
9+	73	74	70	73	75	74	71	70	72	72	73	74	78	74	78	69
10+	74	75	72	78	78	78	74	72	79	76	77	78	81	79	78	74
11+	81	82	78	82	82	83	80	78	81	80	80	83	86	82	86	81
12+	89	86	83	87	87	87	82	83	85	86	86	87	90	86	90	87
13+	90	91	88	90	90	89	87	87	88	88	88	91	92	90	90	88

Item	61	62	63	65	66	67	68	69	70	71	72	73	74	75	76	77
Age																
6+	52	50	—	—	—	—	—	—	—	—	—	—	—	—	—	—
7+	58	54	58	62	65	63	67	66	—	63	57	61	59	59	62	63
8+	63	61	62	68	71	66	71	72	65	70	66	67	67	67	68	68
9+	69	68	69	74	76	71	74	76	72	75	73	73	74	74	74	73
10+	73	74	73	77	80	74	79	79	76	80	79	79	78	78	79	77
11+	80	78	78	82	84	80	84	82	82	85	83	83	84	84	84	83
12+	89	84	83	86	87	84	86	86	86	86	86	85	86	86	86	87
13+	89	87	84	88	90	89	90	90	89	90	90	90	90	90	91	90

Item	78	79	80	81	82	83	84	85	86	87	88	89	90
Age													
6+	—	—	—	—	—	—	—	—	—	—	60	—	—
7+	—	62	62	65	67	69	69	65	—	60	64	—	66
8+	67	68	67	71	71	71	76	73	71	66	72	67	71
9+	74	75	72	74	75	76	79	77	76	74	79	76	76
10+	79	79	77	78	79	81	82	81	81	78	83	80	80
11+	83	82	82	83	83	84	84	85	82	83	87	84	84
12+	85	85	87	87	87	86	87	86	85	86	89	88	88
13+	91	89	88	89	90	90	89	89	89	89	90	91	87

APPENDIX II

n	t	M (6)	M (7)	M (8)	M (9)	M (10)	M (11)	M (12)	M (13)
1	67	74·6	85·0	87·8	92·3	90·2	99·4	102·7	104·1
2	73	73·4	86·0	85·1	88·3	91·3	100·0	105·2	109·4
3	71·5	73·4	83·0	84·1	88·6	90·6	103·7	108·8	110·5
4	72	71·9	83·2	84·5	88·2	91·7	102·7	108·3	110·4
5	75	67·4	79·1	84·9	92·2	95·5	103·0	105·8	110·9
6	71	72·6	81·6	85·3	88·7	90·6	102·3	109·4	111·3
7	71·5	72·5	81·2	84·8	88·9	90·6	102·5	108·4	109·5
8	77	72·0	84·2	91·6	95·2	95·2	98·9	98·7	101·4
9	76	69·7	79·0	85·9	90·5	93·9	102·5	108·7	109·0
10	75	67·6	76·2	83·9	91·7	94·2	102·9	109·5	113·1
11	76	67·1	76·7	85·7	92·5	95·0	103·6	107·8	110·7
12	82	68·3	79·5	87·3	92·5	95·0	100·7	103·2	110·5
13	78·5	65·0	75·4	84·2	92·3	96·2	104·3	111·6	110·0
14	79·5	65·5	75·0	84·5	92·5	97·2	102·2	110·8	112·2
15	82	67·2	77·2	86·0	93·2	96·6	100·9	106·5	109·9
16	80·5	64·3	75·7	86·3	93·0	96·8	104·6	110·0	109·7
17	79·5	63·8	75·5	84·2	93·3	96·2	102·6	112·5	110·5
18	74	74·5	84·3	87·6	91·1	94·1	97·3	102·4	106·8
19	69	71·1	87·0	88·9	98·5	98·3	94·2	107·2	99·3
20	74	70·5	80·2	85·0	92·9	97·0	102·4	101·6	111·6
21	71	65·9	76·4	84·8	92·7	96·8	105·9	107·6	107·6
22	79	71·7	84·1	89·4	92·9	96·0	101·0	101·0	102·6
23	68	66·6	77·4	85·4	94·5	95·7	103·9	107·2	107·8
24	71·5	69·1	79·7	87·4	94·5	93·5	103·0	105·0	107·0
25	71·5	67·4	78·1	86·7	93·7	94·9	99·7	105·0	112·3
26	77·5	70·8	81·6	88·7	93·5	95·2	101·4	103·4	106·4
27	85·5	77·2	84·4	86·4	91·5	92·9	98·2	102·8	104·7
28	89·5	76·6	86·5	88·3	94·1	95·0	96·5	98·5	105·1
29	69·5	64·9	78·0	87·8	95·7	98·1	103·6	106·6	105·1
30	72·5	66·3	77·6	86·9	93·2	96·9	105·4	107·5	106·3
31	77·5	67·8	78·7	88·1	94·2	96·4	103·2	104·4	107·5
32	86·5	67·7	79·0	87·1	93·1	95·9	101·0	105·9	110·1
33	86	—	81·0	87·3	91·8	94·9	100·9	105·2	110·3
34	92	69·7	77·9	83·7	90·1	94·8	102·3	106·4	112·7
35	91	68·0	76·9	83·1	90·8	95·6	103·8	110·6	111·0
36	84·5	69·2	79·9	87·0	94·0	93·2	101·3	105·6	107·8
37	90	68·8	76·2	81·2	88·6	94·6	102·1	110·6	115·4
38	96	74·2	80·3	85·8	92·4	94·8	99·4	102·9	107·2
39	102·5	76·6	84·5	89·0	92·6	95·1	97·7	99·8	103·0
40	102·5	—	77·2	83·5	90·3	94·7	102·5	108·1	112·4
41	93·5	72·0	80·4	85·5	91·3	95·9	100·2	104·4	108·6
42	85	68·2	74·9	81·6	91·8	92·5	102·2	111·2	113·1
43	90	69·7	78·4	84·1	91·8	93·3	102·3	106·7	112·2
44	92	69·6	77·9	83·1	90·8	94·0	102·1	108·7	114·7
45	91·5	69·9	76·8	84·3	92·0	95·3	104·0	110·7	107·1
46	92·5	68·4	77·8	83·8	91·0	97·4	102·9	107·9	111·6
47	94·5	—	78·3	84·5	91·9	96·8	100·8	106·6	109·9
48	96·5	—	77·2	84·3	91·8	96·5	102·8	107·5	110·2
49	95	70·6	77·6	84·1	90·9	94·9	101·6	106·7	112·9
50	91·5	—	76·0	83·4	91·1	95·3	102·6	106·4	112·8
51	91·5	68·0	77·0	83·5	91·1	94·0	102·2	109·3	113·1
52	97·5	—	79·3	86·0	90·0	96·9	101·5	106·4	110·3
53	93·5	—	76·3	83·6	91·5	96·6	102·0	109·7	111·8
54	95·5	—	77·4	84·1	92·2	96·4	101·1	109·3	111·1
55	108·5	—	—	84·7	91·4	95·5	100·6	105·8	111·2
56	101·5	—	78·7	85·9	91·9	95·9	101·5	106·5	109·2
57	113	—	—	86·9	91·4	96·2	100·3	105·3	109·5

APPENDIX II (*contd.*)

n	t	M (6)	M (7)	M (8)	M (9)	M (10)	M (11)	M (12)	M (13)
58	108	—	79·5	86·2	92·6	92·9	102·4	107·5	108·6
59	112	—	—	88·2	93·0	97·2	100·2	103·6	108·3
60	100	—	—	79·0	88·9	94·9	102·9	111·7	112·0
61	104	—	—	80·6	88·8	93·8	101·9	113·2	113·2
62	102	—	—	79·9	88·5	95·0	102·0	110·9	113·6
63	109·5	—	—	82·6	90·3	95·6	102·4	109·4	110·2
64	122·5	—	—	—	90·6	94·1	100·8	106·2	113·3
65	113	—	—	84·5	91·6	95·7	102·0	107·2	109·9
66	120	—	—	—	93·0	97·2	102·1	105·4	108·1
67	122	—	—	—	—	93·2	101·1	106·5	112·0
68	130	—	—	—	—	95·9	102·3	105·2	110·5
69	127·5	—	—	—	—	97·0	100·9	106·3	109·5
70	113·5	—	—	—	89·7	94·0	102·5	108·2	111·1
71	113·5	—	—	84·7	90·6	97·3	102·8	104·8	110·1
72	112	—	—	—	89·6	96·8	101·8	105·7	111·6
73	112	—	—	82·8	90·3	96·9	102·3	105·7	111·8
74	114	—	—	—	89·8	96·0	102·8	104·9	110·6
75	115	—	—	—	90·2	96·6	102·2	105·1	110·4
76	113·5	—	—	83·6	90·3	96·9	102·8	105·8	111·2
77	117·5	—	—	—	90·1	95·8	101·8	106·3	111·4
78	123	—	—	—	90·9	96·8	102·0	104·4	111·3
79	121	—	—	—	92·1	96·8	101·6	104·7	110·5
80	111·5	—	—	83·7	90·1	95·9	102·2	108·4	109·9
81	115	—	—	86·4	90·6	96·0	101·4	106·1	109·0
82	122	—	—	—	91·4	96·6	102·0	105·5	109·9
83	129·5	—	—	—	—	97·9	101·9	105·0	109·2
84	131·5	—	—	—	—	98·8	101·8	105·3	108·1
85	122·5	—	—	—	92·6	98·4	102·5	103·5	107·9
86	130	—	—	—	—	99·6	100·0	104·5	109·6
87	112	—	—	—	90·9	96·3	101·7	105·6	110·0
88	110·5	—	—	84·9	93·5	97·8	102·2	104·6	106·1
89	127	—	—	—	—	96·8	101·3	105·2	109·9
90	123·5	—	—	—	92·5	97·2	102·4	106·7	106·6
91	141·5	—	—	—	—	—	—	—	109·9
92	120·5	—	—	—	92·1	95·1	101·1	106·3	111·2
93	125·5	—	—	—	—	94·5	100·9	106·2	112·3
94	125·5	—	—	—	—	93·2	101·5	105·9	112·8
95	128	—	—	—	—	96·0	100·6	105·8	111·1
96	139·5	—	—	—	—	—	—	104·6	111·5
97	145·5	—	—	—	—	—	—	—	109·9
98	142	—	—	—	—	—	—	105·4	110·6
99	140·5	—	—	—	—	—	—	105·9	110·4
100	138	—	—	—	—	—	103·2	103·5	111·1
Totals		3207·3	4443·9	5965·7	7700·1	8971·2	9664·5	10425·0	10978·6
No. of items		46	56	70	84	94	95	98	100
Means		69·7	79·4	85·2	91·7	95·4	101·7	106·4	109·8
Mean variation		2·66	2·56	1·76	1·39	1·32	1·20	2·14	1·99
Greatest variation (positive)		7·5	7·6	6·4	6·8	4·2	4·2	6·8	5·6
Greatest variation (negative)		5·9	4·4	6·2	3·5	5·2	7·5	7·9	10·5

Mean variation over the whole 643 entries 1·80

APPENDIX III

Table giving $100e^{-e^{-t}}$ (percentage development), against t (isochronic age). The first column gives isochronic age, the second percentage development, while the third gives the difference between the entry in the second column and the entry immediately below it.

t	$100e^{-e^{-t}}$	Diff.	t	$100e^{-e^{-t}}$	Diff.	t	$100e^{-e^{-t}}$	Diff.
−2·0	0·05	0·07	0	36·79	3·67	2·0	87·34	1·13
−1·9	0·12	0·12	0·1	40·46	3·64	2·1	88·47	1·05
−1·8	0·24	0·18	0·2	44·10	3·56	2·2	89·52	0·92
−1·7	0·42	0·29	0·3	47·66	3·48	2·3	90·44	0·89
−1·6	0·71	0·42	0·4	51·14	3·39	2·4	91·33	0·77
−1·5	1·13	0·60	0·5	54·53	3·24	2·5	92·10	0·73
−1·4	1·73	0·82	0·6	57·77	3·08	2·6	92·83	0·67
−1·3	2·55	1·06	0·7	60·85	2·96	2·7	93·50	0·60
−1·2	3·61	1·34	0·8	63·81	2·78	2·8	94·10	0·54
−1·1	4·95	1·65	0·9	66·59	2·62	2·9	94·64	0·51
−1·0	6·60	1·95	1·0	69·21	2·48	3·0	95·15	0·46
−0·9	8·55	2·25	1·1	71·69	2·30	3·1	95·61	0·40
−0·8	10·80	2·55	1·2	73·99	2·15	3·2	96·01	0·37
−0·7	13·35	2·82	1·3	76·14	2·00	3·3	96·38	0·34
−0·6	16·17	3·06	1·4	78·14	1·84	3·4	96·72	0·31
−0·5	19·23	3·27	1·5	79·98	1·74	3·5	97·03	0·26
−0·4	22·50	3·43	1·6	81·72	1·60	3·6	97·29	0·28
−0·3	25·93	3·55	1·7	83·32	1·44	3·7	97·57	0·22
−0·2	29·48	3·64	1·8	84·76	1·34	3·8	97·79	0·21
−0·1	33·12	3·67	1·9	86·10	1·24	3·9	98·00	0·19
						4·0	98·19	

THE FUNDAMENTAL DATA

P (6), *P* (7), etc. are the percentage of 6-year-olds, 7-year-olds, etc. answering the various items correctly. The *X*'s are the σ values corresponding to the *P*'s.

Item	P (6)	X (6)	X^2 (6)	P (7)	X (7)	X^2 (7)	P (8)	X (8)	X^2 (8)	P (9)	X (9)	X^2 (9)
1	71·67	−0·57	0·325	88·00	−1·175	1·381	90·69	−1·32	1·742	93·55	−1·52	2·310
2	51·18	−0·03	0·001	80·07	−0·84	0·706	78·47	−0·79	0·624	83·24	−0·96	0·922
3	55·62	−0·14	0·020	78·07	−0·77	0·593	79·68	−0·83	0·789	85·69	−1·07	1·145
4	49·78	+0·005	0·000	77·36	−0·75	0·563	79·46	−0·82	0·672	84·64	−1·02	1·040
5	26·78	+0·62	0·384	61·28	−0·29	0·084	74·23	−0·65	0·423	84·99	−1·04	1·082
6	54·95	−0·12	0·014	76·40	−0·72	0·518	82·32	−0·93	0·865	86·65	−1·11	1·232
7	52·96	−0·075	0·006	74·57	−0·66	0·436	80·78	−0·87	0·757	86·15	−1·09	1·188
8	35·21	+0·38	0·144	68·01	−0·47	0·221	80·95	−0·88	0·774	85·55	−1·06	1·124
9	30·77	+0·50	0·25	58·49	−0·21	0·044	73·79	−0·64	0·410	81·34	−0·89	0·792
10	27·00	+0·61	0·372	53·69	−0·09	0·008	72·30	−0·59	0·348	84·45	−1·01	1·020
11	23·15	+0·73	0·533	52·07	−0·05	0·003	73·51	−0·63	0·397	83·99	−0·99	0·980
12	13·09	+1·12	1·254	43·38	+0·17	0·029	63·38	−0·34	0·116	73·63	−0·63	0·397
13	12·65	+1·16	1·346	41·05	+0·23	0·053	64·37	−0·37	0·137	79·64	−0·83	0·689
14	11·69	+1·19	1·416	37·22	+0·33	0·109	62·83	−0·33	0·109	78·29	−0·78	0·608
15	11·02	+1·23	1·513	36·06	+0·36	0·130	60·24	−0·26	0·068	74·88	−0·67	0·449
16	8·14	+1·40	1·960	36·32	+0·35	0·123	64·37	−0·37	0·137	77·35	−0·75	0·563
17	8·51	+1·37	1·877	38·59	+0·29	0·084	62·28	−0·31	0·096	79·28	−0·82	0·672
18	51·48	−0·04	0·002	75·09	−0·68	0·462	80·40	−0·86	0·740	85·10	−1·04	1·082
19	56·88	−0·17	0·029	87·54	−1·15	1·323	89·26	−1·24	1·538	95·20	−1·665	2·772
20	39·20	+0·27	0·073	66·53	−0·43	0·185	76·27	−0·715	0·511	87·05	−1·13	1·277
21	33·51	+0·43	0·185	65·16	−0·39	0·152	81·61	−0·90	0·810	90·29	−1·30	1·690
22	28·70	+0·56	0·314	63·17	−0·34	0·116	74·23	−0·65	0·423	79·78	−0·83	0·689
23	45·27	+0·12	0·014	75·02	−0·675	0·456	87·06	−1·13	1·277	94·04	−1·56	2·434
24	41·49	+0·21	0·044	71·64	−0·57	0·325	84·31	−1·01	1·020	91·21	−1·35	1·823
25	36·54	+0·34	0·116	67·89	−0·465	0·216	83·48	−0·97	0·941	90·57	−1·315	1·729
26	29·96	+0·53	0·281	60·89	−0·28	0·078	75·94	−0·705	0·497	82·79	−0·95	0·903
27	27·88	+0·59	0·348	47·14	+0·07	0·005	52·20	−0·055	0·003	64·04	−0·36	0·130
28	17·68	+0·93	0·865	42·61	+0·19	0·036	47·19	+0·07	0·005	60·63	−0·27	0·073
29	34·84	+0·39	0·152	73·02	−0·61	0·372	87·72	−1·16	1·346	93·59	−1·52	2·310
30	29·88	+0·53	0·281	64·07	−0·36	0·130	81·99	−0·915	0·837	89·04	−1·23	1·513
31	21·45	+0·79	0·624	53·18	−0·08	0·006	74·94	−0·67	0·449	83·73	−0·98	0·960
32	6·07	+1·55	2·403	29·95	+0·53	0·281	51·49	−0·04	0·002	65·28	−0·39	0·152
33	1·24	+2·24	5·018	36·70	+0·34	0·116	53·41	−0·085	0·007	63·63	−0·35	0·123
34	3·85	+1·77	3·133	15·82	+1·00	1·000	29·02	+0·55	0·303	45·32	+0·12	0·014
35	3·03	+1·88	3·534	15·44	+1·02	1·040	28·74	+0·53	0·281	49·72	+0·01	0·000
36	10·95	+1·23	1·513	37·68	+0·31	0·096	56·55	−0·165	0·027	71·23	−0·56	0·314
37	4·36	+1·71	2·924	15·89	+1·00	1·000	27·31	+0·60	0·360	46·42	+0·09	0·008
38	5·10	+1·64	2·690	13·88	+1·09	1·188	25·55	+0·66	0·436	41·22	+0·22	0·048
39	3·03	+1·88	3·534	11·68	+1·19	1·416	19·99	+0·84	0·706	27·71	+0·59	0·348
40	0·44	+2·62	6·864	3·44	+1·82	3·312	10·19	+1·27	1·613	22·57	+0·75	0·563
41	4·88	+1·66	2·756	18·31	+0·90	0·810	30·18	+0·52	0·270	44·82	+0·13	0·017
42	8·58	+1·37	1·877	22·63	+0·75	0·563	40·91	+0·23	0·053	65·93	−0·41	0·168
43	5·25	+1·62	2·624	20·49	+0·82	0·672	34·75	+0·39	0·152	54·52	−0·11	0·012
44	3·70	+1·79	3·204	15·89	+1·00	1·000	27·64	+0·59	0·348	47·32	+0·07	0·005
45	4·29	+1·72	2·958	14·14	+1·07	1·145	31·00	+0·47	0·221	51·17	−0·03	0·001
46	2·51	+1·96	3·842	14·72	+1·05	1·103	28·47	+0·57	0·325	46·37	+0·09	0·008
47	1·78	+2·10	4·410	11·54	+1·15	1·323	25·55	+0·66	0·436	43·97	+0·15	0·023
48	1·63	+2·14	4·580	8·17	+1·39	1·932	21·20	+0·80	0·640	38·96	+0·28	0·078
49	2·76	+1·92	3·686	10·64	+1·25	1·563	23·62	+0·72	0·518	40·32	+0·245	0·060
50	2·22	+2·01	4·040	11·80	+1·13	1·277	29·63	+0·535	0·286	49·27	+0·02	0·000

THE FUNDAMENTAL DATA

P (6), P (7), etc. are the percentage of 6-year-olds, 7-year-olds, etc. answering the various items correctly. The X's are the σ values corresponding to the P's.

P (10)	X (10)	X^2 (10)	P (11)	X (11)	X^2 (11)	P (12)	X (12)	X^2 (12)	P (13)	X (13)	X^2 (13)
92·40	−1·43	2·045	96·48	−1·81	3·276	97·34	−1·93	3·725	97·63	−1·98	3·920
86·76	−1·115	1·243	93·31	−1·50	2·250	95·57	−1·70	2·890	96·81	−1·85	3·423
87·97	−1·17	1·369	95·69	−1·725	2·976	97·16	−1·905	3·629	97·50	−1·96	3·842
88·31	−1·19	1·416	95·16	−1·66	2·756	96·88	−1·86	3·460	97·34	−1·93	3·725
88·26	−1·19	1·416	93·49	−1·51	2·280	94·77	−1·62	2·624	96·40	−1·80	3·240
88·54	−1·20	1·440	95·51	−1·70	2·890	97·43	−1·95	3·803	97·77	−2·01	4·040
87·86	−1·17	1·369	95·33	−1·68	2·822	97·06	−1·89	3·572	97·34	−1·93	3·725
85·55	−1·06	1·124	89·08	−1·23	1·513	88·40	−1·22	1·488	91·01	−1·34	1·796
85·43	−1·055	1·113	92·43	−1·435	2·059	95·23	−1·67	2·789	95·37	−1·68	2·822
87·10	−1·13	1·277	93·39	−1·505	2·265	96·01	−1·75	3·063	96·94	−1·87	3·497
86·70	−1·11	1·232	93·05	−1·48	2·190	94·95	−1·64	2·690	95·94	−1·74	3·028
77·55	−0·76	0·578	84·77	−1·03	1·061	87·25	−1·14	1·300	92·32	−1·43	2·045
84·53	−1·02	1·040	91·55	−1·375	1·891	95·05	−1·65	2·723	94·39	−1·59	2·528
84·28	−1·01	1·020	89·17	−1·235	1·525	94·22	−1·57	2·465	94·73	−1·62	2·624
79·97	−0·84	0·706	85·03	−1·04	1·082	89·91	−1·28	1·638	92·12	−1·41	1·988
82·50	−0·935	0·874	89·96	−1·28	1·638	93·21	−1·49	2·220	93·09	−1·48	2·190
83·19	−0·96	0·941	89·44	−1·25	1·563	94·86	−1·63	2·657	94·10	−1·56	2·434
88·14	−1·18	1·392	90·84	−1·33	1·769	93·85	−1·54	2·372	95·59	−1·705	2·907
95·11	−1·655	2·739	93·13	−1·485	2·205	97·61	−1·98	3·920	95·50	−1·695	2·873
90·67	−1·32	1·742	93·84	−1·54	2·372	93·49	−1·51	2·280	96·94	−1·87	3·497
83·03	−1·48	2·190	96·66	−1·83	3·349	97·06	−1·89	3·572	97·07	−1·89	3·572
83·59	−0·98	0·960	88·64	−1·21	1·464	88·61	−1·21	1·464	89·95	−1·28	1·638
94·59	−1·61	2·592	97·36	−1·92	3·686	97·89	−2·03	4·121	97·97	−2·05	4·203
90·56	−1·31	1·716	95·51	−1·70	2·890	96·15	−1·77	3·133	96·74	−1·845	3·404
91·48	−1·37	1·877	94·19	−1·57	2·465	96·15	−1·77	3·133	97·83	−2·02	4·080
84·86	−1·03	1·061	90·49	−1·31	1·716	91·83	−1·39	1·932	93·42	−1·51	2·280
66·95	−0·44	0·194	76·54	−0·72	0·518	82·39	−0·93	0·865	84·59	−1·02	1·040
62·75	−0·325	0·106	65·58	−0·40	0·160	69·36	−0·51	0·260	79·54	−0·825	0·681
94·70	−1·62	2·624	96·66	−1·83	3·349	97·34	−1·93	3·725	97·01	−1·88	3·534
91·88	−1·40	1·960	95·86	−1·735	3·010	96·51	−1·81	3·276	96·12	−1·765	3·115
86·30	−1·09	1·188	91·64	−1·38	1·904	92·39	−1·43	2·045	93·91	−1·55	2·403
70·70	−0·545	0·297	78·69	−0·80	0·640	84·49	−1·015	1·030	88·24	−1·19	1·416
69·83	−0·52	0·270	79·31	−0·82	0·672	84·40	−1·01	1·020	88·88	−1·22	1·488
56·53	−0·165	0·027	71·21	−0·56	0·314	77·34	−0·75	0·563	84·59	−1·02	1·040
60·74	−0·27	0·073	75·35	−0·685	0·469	83·67	−0·98	0·960	84·16	−1·00	1·000
69·72	−0·52	0·270	82·13	−0·92	0·846	86·61	−1·11	1·232	88·51	−1·20	1·440
60·79	−0·27	0·073	74·38	−0·655	0·429	84·95	−1·035	1·071	88·94	−1·22	1·488
47·09	+0·07	0·005	57·39	−0·19	0·036	64·40	−0·37	0·137	71·99	−0·58	0·336
33·33	+0·43	0·185	39·44	+0·27	0·073	43·94	+0·15	0·023	51·28	−0·03	0·001
32·47	+0·455	0·207	49·91	+0·00	0·000	61·28	−0·29	0·084	68·90	−0·49	0·240
55·50	−0·14	0·020	64·52	−0·37	0·137	71·93	−0·58	0·336	77·85	−0·77	0·593
67·24	−0·45	0·203	82·39	−0·93	0·865	90·54	−1·31	1·716	91·62	−1·38	1·904
58·03	−0·20	0·040	74·82	−0·67	0·449	80·83	−0·87	0·757	86·38	−1·10	1·210
54·86	−0·12	0·014	70·95	−0·55	0·303	80·37	−0·85	0·723	86·52	−1·10	1·210
58·89	−0·225	0·051	74·91	−0·67	0·449	83·21	−0·96	0·922	79·12	−0·81	0·656
61·02	−0·28	0·078	71·21	−0·56	0·314	78·62	−0·79	0·624	82·99	−0·95	0·903
55·21	−0·13	0·017	63·82	−0·35	0·123	73·67	−0·63	0·397	78·16	−0·78	0·608
50·08	+0·00	0·000	63·47	−0·34	0·116	71·47	−0·57	0·325	75·53	−0·69	0·476
49·80	+0·005	0·000	64·00	−0·36	0·130	72·94	−0·61	0·372	80·93	−0·875	0·766
58·66	−0·22	0·048	72·62	−0·60	0·360	78·26	−0·78	0·608	85·33	−1·05	1·103

THE FUNDAMENTAL DATA (*contd.*)

Item	P (6)	X (6)	X^2 (6)	P (7)	X (7)	X^2 (7)	P (8)	X (8)	X^2 (8)	P (9)	X (9)	X^2 (9)
51	2·76	+1·92	3·686	14·72	+1·05	1·103	29·90	+0·53	0·281	48·97	+0·025	0·001
52	1·63	+2·14	4·580	9·99	+1·28	1·638	23·02	+0·74	0·548	32·36	+0·46	0·212
53	1·41	+2·20	4·840	10·44	+1·26	1·588	25·88	+0·65	0·423	45·12	+0·12	0·014
54	1·85	+2·09	4·368	9·60	+1·305	1·181	22·63	+0·75	0·563	41·97	+0·20	0·040
55	0·22	+2·85	8·123	2·01	+2·05	4·203	5·95	+1·56	2·434	14·86	+1·04	1·081
56	0·74	+2·44	5·954	5·32	+1·615	2·608	15·69	+1·01	1·020	28·06	+0·58	0·336
57	0·30	+2·75	7·563	1·62	+2·14	4·580	4·79	+1·67	2·789	9·31	+1·32	1·742
58	0·15	+2·97	8·821	2·33	+1·99	3·960	7·93	+1·41	1·988	17·76	+0·92	0·846
59	0·52	+2·56	6·554	2·21	+2·01	4·040	6·72	+1·50	2·250	12·71	+1·14	1·300
60	0·22	+2·85	8·123	2·27	+2·00	4·000	6·88	+1·485	2·205	24·51	+0·69	0·476
61	0·074	+3·16	9·986	1·10	+2·29	5·244	5·29	+1·62	2·624	17·06	+0·95	0·903
62	0·074	+3·16	9·986	0·58	+2·525	6·376	6·11	+1·55	2·403	20·06	+0·84	0·706
63	0	—	—	0·84	+2·39	5·712	3·52	+1·81	3·276	11·91	+1·18	1·392
64	0	—	—	0·39	+2·66	7·076	0·99	+2·33	5·429	2·56	+1·95	3·803
65	0	—	—	0·52	+2·56	6·554	3·08	+1·87	3·497	9·66	+1·30	1·690
66	0	—	—	0·19	+2·895	8·381	1·43	+2·19	4·796	5·26	+1·62	2·624
67	0	—	—	0·13	+3·015	9·090	0·50	+2·575	6·631	2·15	+2·025	4·101
68	0	—	—	0·064	+3·22	10·368	0·33	+2·72	7·398	0·76	+2·43	5·905
69	0	—	—	0·064	+3·22	10·368	0·55	+2·54	6·452	1·75	+2·11	4·452
70	0	—	—	0	—	—	1·32	+2·22	4·928	7·26	+1·46	2·132
71	0	—	—	0·39	+2·66	7·076	2·92	+1·89	3·572	7·95	+1·41	1·988
72	0	—	—	0·064	+3·22	10·368	2·09	+2·035	4·141	8·21	+1·39	1·932
73	0	—	—	0·39	+2·66	7·076	2·48	+1·96	3·842	8·96	+1·34	1·796
74	0	—	—	0·064	+3·22	10·368	1·38	+2·20	4·840	6·71	+1·50	2·250
75	0	—	—	0·13	+3·015	9·090	1·76	+2·105	4·431	6·21	+1·54	2·372
76	0	—	—	0·26	+2·795	7·812	2·37	+1·98	3·920	7·61	+1·43	2·045
77	0	—	—	0·19	+2·895	8·381	1·49	+2·17	4·709	4·51	+1·695	2·873
78	0	—	—	0	—	—	0·33	+2·72	7·398	2·50	+1·96	3·842
79	0	—	—	0·064	+3·22	10·368	0·61	+2·505	6·275	4·01	+1·75	3·063
80	0	—	—	0·79	+2·415	5·832	3·30	+1·84	3·386	9·26	+1·325	1·756
81	0	—	—	0·65	+2·485	6·175	3·25	+1·845	3·404	6·66	+1·50	2·250
82	0	—	—	0·26	+2·795	7·812	1·16	+2·27	5·153	3·11	+1·865	3·478
83	0	—	—	0·13	+3·015	9·090	0·28	+2·77	7·673	1·51	+2·17	4·709
84	0	—	—	0·064	+3·220	10·368	0·77	+2·42	5·856	2·01	+2·05	4·203
85	0	—	—	0·13	+3·015	9·090	1·71	+2·12	4·494	3·96	+1·79	3·204
86	0	—	—	0	—	—	0·28	+2·77	7·673	1·26	+2·24	5·018
87	0	—	—	0·19	+2·895	8·381	1·71	+2·12	4·494	9·76	+1·295	1·677
88	0·074	+3·16	9·986	0·58	+2·525	6·376	4·68	+1·68	2·822	15·56	+1·01	1·020
89	0	—	—	0	—	—	0·11	+3·07	9·425	1·91	+2·07	4·285
90	0	—	—	0·13	+3·015	9·090	0·77	+2·42	5·856	3·16	+1·86	3·460
91	0	—	—	0	—	—	0·17	+2·92	8·526	0·46	+2·605	6·786
92	0	—	—	0·26	+2·795	7·812	0·94	+2·35	5·523	4·31	+1·715	2·941
93	0	—	—	0·06	+3·220	10·368	0·33	+2·72	7·398	1·71	+2·12	4·494
94	0	—	—	0·13	+3·015	9·090	0·17	+2·92	8·526	1·31	+2·22	4·928
95	0	—	—	0·13	+3·015	9·090	0·99	+2·33	5·429	1·46	+2·18	4·752
96	0	—	—	0·06	+3·220	10·368	0·50	+2·575	6·631	1·10	+2·29	5·244
97	0	—	—	0	—	—	0	—	—	0·15	+2·965	8·791
98	0	—	—	0	—	—	0·06	+3·220	10·368	0	—	—
99	0	—	—	0	—	—	0	—	—	0·15	+2·965	8·791
100	0	—	—	0·13	+3·015	9·090	0·06	+3·22	10·368	0·30	+2·75	7·563
Total (+)	—	85·065	—	—	129·620	—	—	109·880	—	—	80·125	—
Total (−)	—	1·145	—	—	12·055	—	—	22·200	—	—	33·590	—
Total	1124·622	+83·920	172·903	2312·024	+117·565	326·425	3124·63	+87·680	253·913	3985·96	+46·535	182·807

THE FUNDAMENTAL DATA (*contd.*)

P (10)	X (10)	X^2 (10)	P (11)	X (11)	X^2 (11)	P (12)	X (12)	X^2 (12)	P (13)	X (13)	X^2 (13)
55·90	−0·15	0·023	72·01	−0·58	0·336	81·63	−0·90	0·810	85·47	−1·06	1·124
48·65	+0·035	0·001	58·71	−0·22	0·048	67·71	−0·46	0·212	74·23	−0·65	0·423
57·28	−0·180	0·032	67·69	−0·46	0·212	79·45	−0·82	0·672	81·49	−0·90	0·810
52·10	−0·05	0·003	62·15	−0·31	0·096	75·96	−0·705	0·497	78·16	−0·78	0·608
22·51	+0·755	0·570	33·19	+0·435	0·189	44·59	+0·14	0·020	55·35	−0·135	0·018
37·42	+0·32	0·102	50·18	+0·00	0·000	60·18	−0·26	0·068	65·03	−0·39	0·152
16·52	+0·97	0·941	24·29	+0·70	0·49	34·59	+0·40	0·160	43·05	+0·175	0·031
18·31	+0·90	0·810	38·29	+0·30	0·09	48·72	+0·03	0·001	51·41	−0·035	0·001
19·98	+0·84	0·706	25·88	+0·65	0·423	32·85	+0·445	0·198	42·57	+0·19	0·036
38·34	+0·295	0·087	56·25	−0·16	0·026	71·83	−0·58	0·336	72·41	−0·595	0·354
27·52	+0·60	0·360	45·60	+0·11	0·012	67·25	−0·45	0·203	67·23	−0·45	0·203
34·08	+0·41	0·168	50·00	0·00	0·000	65·69	−0·40	0·160	71·71	−0·57	0·325
20·84	+0·81	0·656	35·12	+0·38	0·144	49·82	+0·005	0·000	51·41	−0·035	0·001
4·61	+1·685	2·839	11·53	+1·20	1·440	19·72	+0·85	0·723	32·61	+0·45	0·203
15·95	+1·00	1·000	27·73	+0·59	0·348	38·35	+0·295	0·087	44·16	+0·15	0·023
9·56	+1·31	1·716	16·55	+0·97	0·941	22·02	+0·77	0·593	26·96	+0·61	0·372
4·26	+1·72	2·958	12·41	+1·15	1·323	21·01	+0·81	0·656	31·17	+0·49	0·240
2·42	+1·97	3·881	6·60	+1·51	2·280	9·27	+1·32	1·742	16·38	+0·98	0·960
4·03	+1·75	3·063	7·13	+1·47	2·161	13·39	+1·11	1·23	18·24	+0·91	0·828
12·26	+1·16	1·346	27·64	+0·59	0·348	39·17	+0·275	0·076	45·20	+0·12	0·014
17·96	+0·92	0·846	28·26	+0·575	0·331	32·39	+0·46	0·212	43·40	+0·17	0·029
19·11	+0·87	0·757	28·96	+0·555	0·308	37·25	+0·325	0·106	49·20	+0·02	0·000
19·34	+0·865	0·748	29·92	+0·53	0·281	37·25	+0·325	0·106	49·55	+0·01	0·000
14·80	+1·045	1·092	27·55	+0·60	0·360	31·60	+0·48	0·230	43·26	+0·17	0·029
14·39	+1·06	1·124	24·38	+0·69	0·476	30·28	+0·52	0·270	40·84	+0·23	0·053
17·16	+0·95	0·903	28·52	+0·57	0·325	34·59	+0·40	0·160	45·46	+0·11	0·012
10·25	+1·27	1·613	19·45	+0·86	0·740	27·98	+0·58	0·336	38·15	+0·30	0·090
6·51	+1·51	2·280	12·50	+1·15	1·323	16·24	+0·985	0·970	27·99	+0·58	0·336
8·18	+1·39	1·932	14·52	+1·06	1·124	19·36	+0·865	0·748	29·99	+0·525	0·276
18·42	+0·90	0·810	30·98	+0·50	0·250	43·58	+0·16	0·026	46·92	+0·08	0·006
13·53	+1·10	1·210	22·80	+0·745	0·555	32·29	+0·46	0·212	38·35	+0·30	0·090
7·20	+1·46	2·132	13·73	+1·09	1·188	19·36	+0·865	0·748	27·09	+0·61	0·372
3·68	+1·79	3·204	6·51	+1·51	2·280	9·72	+1·30	1·690	15·20	+1·03	1·061
3·28	+1·84	3·386	5·19	+1·63	2·657	8·35	+1·38	1·904	11·33	+1·21	1·464
8·52	+1·37	1·877	14·08	+1·08	1·166	15·41	+1·02	1·040	22·74	+0·75	0·563
4·43	+1·70	2·890	4·75	+1·67	2·789	8·716	+1·36	1·850	15·07	+1·03	1·061
18·37	+0·90	0·810	28·87	+0·56	0·314	36·97	+0·33	0·109	46·16	+0·10	0·010
23·60	+0·72	0·518	32·83	+0·445	0·198	37·98	+0·305	0·093	40·91	+0·23	0·027
4·09	+1·74	3·028	7·92	+1·41	1·988	12·39	+1·155	1·334	19·35	+0·865	0·748
6·56	+1·51	2·280	12·50	+1·15	1·323	19·27	+0·87	0·757	18·87	+0·88	0·774
0·287	+2·76	7·618	0·79	+2·415	5·832	1·10	+2·29	5·244	5·46	+1·60	2·560
6·74	+1·495	2·235	14·17	+1·07	1·145	22·94	+0·74	0·548	32·21	+0·46	0·212
3·40	+1·825	3·331	8·71	+1·36	1·850	15·60	+1·01	1·020	25·70	+0·65	0·423
2·71	+1·925	3·706	9·27	+1·32	1·676	16·33	+0·98	0·960	26·87	+0·62	0·384
3·22	+1·85	3·423	6·43	+1·52	2·310	12·02	+1·17	1·369	19·00	+0·84	0·706
1·15	+2·27	5·153	1·94	+2·065	4·264	3·22	+1·85	3·423	8·085	+1·40	1·960
0·057	+3·23	10·433	0·53	+2·555	6·528	1·47	+2·18	4·752	3·593	+1·80	3·240
0·172	+2·91	8·468	0·99	+2·33	5·429	2·66	+1·93	3·725	5·667	+1·58	2·496
0·40	+2·65	7·023	1·94	+2·065	4·264	3·48	+1·815	3·294	6·427	+1·52	2·310
0·92	+2·35	5·523	3·08	+1·87	3·497	3·22	+1·85	3·423	8·912	+1·35	1·823
—	65·665	—	—	47·275	—	—	35·88	—	—	25·095	—
—	39·590	—	—	56·165	—	—	69·36	—	—	75·140	—
4492·976	+25·075	156·037	5311·32	−8·890	141·297	5885·066	−33·48	144·837	6388·894	−50·045	137·813